Recovery of singularities in quasilinear biharmonic operator

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The linear Euler-Lagrange equation that arises from vibrating of the beam contains (in the simplest model) derivatives of the 4th and 2nd order

$$u^{(4)}(x) - cu''(x) = p(x),$$

where $u(x)$ denotes the deviation from the equilibrium of the beam at point x and $p(x)$ is the density of the lateral load at x.
If we consider a suspension bridge as a beam of length L with hinged ends then downward deflection is measured by $u(x, t)$ that satisfies the equation of order four (with Navier boundary conditions) :

$$\gamma u_{xxxx}(x, t) + u_{tt}(x, t) = -ku^{+}(x, t) + W + f(x, t),$$

$$u(0, t) = u(L, t) = u_{xx}(0, t) = u_{xx}(L, t) = 0,$$

where γ, W, k are constants, and $f(x, t)$ is the external forcing term.

A multidimensional nonlinear beam-equation (see Gazzola and et al, "Polyharmonic boundary value problems", Springer, 2010) :

$$\partial_t^2 U(x,t) + \Delta^2 U(x,t) + m(x)U(x,t)|U(x,t)|^p = 0,$$

where $p \geq 0$, under time-harmonic assumption $U(x,t) = u(x)e^{-i\omega t}$ leads to the biharmonic equation

$$\Delta^2 u(x) + m(x)u(x)|u(x)|^p = \omega^2 u(x).$$

The wave parameter ω is fixed (in general), but nevertheless we can consider it fixed but very big in order to apply limiting process and appropriate numerical methods. This allows to consider some scattering problems with high frequency for this potential equation.

We deal with quasilinear operators of fourth order of the form

$$H_4 u(x) := \Delta^2 u(x) + \vec{W}(x,|u|)\nabla u(x) + V(x,|u|)u(x), \quad x \in R^n, n = 1,2,3,$$

where $\vec{W}$ and V are complex-valued (in general) and s.t. $V(x,1), \vec{W}(x,1)$ belong to $L^p_{loc}(R^n)$ with p depending on n, and in addition, for any $\rho > 0$

$$|V(x,s_1) - V(x,s_2)| \leq C_\rho \beta_V(x)|s_1 - s_2|,$$

$$|\vec{W}(x,s_1) - \vec{W}(x,s_2)| \leq C_\rho \beta_W(x)|s_1 - s_2|, \quad 0 \leq s_1, s_2 \leq \rho,$$

and both (together with β_V, β_W) have behaviour at the infinity

$$|\vec{W}(x,1)|, \quad |V(x,1)| \leq \frac{C}{|x|^\mu}, \quad |x| > R, \quad \mu > n.$$

These conditions include the power-type nonlinearities of the nonlinear beam equation described above, and most other physically relevant nonlinearities, such as the saturation and sinc nonlinearities.

The scattering problems are connected with the special solutions of the differential equation $H_4 u(x) = k^4 u(x)$, i.e., the solutions in the form

$$u(x,k,\theta) = u_0(x,k,\theta) + u_{sc}(x,k,\theta), \quad u_0(x,k,\theta) = e^{ik(x,\theta)}, \quad \theta \in S^{n-1},$$

where u_0 is the incident (plane) wave with angle θ and the scattered filed u_{sc} satisfies the Sommerfeld radiation conditions

$$r^{\frac{n-1}{2}} \left(\frac{\partial}{\partial r} - ik \right) u_{sc}(x,k,\theta) = o(1), \quad r = |x| \to \infty,$$

$$r^{\frac{n-1}{2}} \left(\frac{\partial}{\partial r} - ik \right) \Delta u_{sc}(x,k,\theta) = o(1), \quad r = |x| \to \infty.$$

These solutions give us the data for inverse problem.

Under the Sommerfeld radiation conditions these solutions ($\equiv$ scattering solutions) are the unique solutions of the following integral equation (analogue of the Lippmann-Schwinger equation for linear Schrödinger operator)

$$u(x,k,\theta) = u_0 - \int_{R^n} G_k^+(|x-y|)\left(\vec{W}(y,|u|)\nabla + V(y,|u|)\right)u(y,k,\theta)\,dy,$$

where G_k^+ is the outgoing fundamental solution of the operator $\Delta^2 - k^4$. It is the kernel of the integral operator $(\Delta^2 - k^4 - i0)^{-1}$ and it is equal to

$$G_k^+(k|x|) = \frac{i}{8k^2}\left(\frac{k}{2\pi|x|}\right)^{\frac{n-2}{2}}\left(H^{(1)}_{\frac{n-2}{2}}(k|x|) + \frac{2i}{\pi}K_{\frac{n-2}{2}}(k|x|)\right), \quad k>0,$$

where $H^{(1)}_\nu$ and K_ν are the Hankel and Macdonald's functions of order ν, respectively. This integral operator maps as follows (due to Agmon's estimates for the operator $-\Delta - k^2$) :

$$\|(\Delta^2 - k^4 - i0)^{-1}f\|_{W^s_{2,-\delta}} \le \frac{C}{k^{3-s}}\|f\|_{L^2_\delta}, \quad s=0,1,2, \quad \delta > \frac{1}{2}, \quad k \ge 1.$$

In the linear case these estimates are enough to prove the unique solvability for such solutions in the spaces $W^1_{2,-\delta}(R^n)$ and obtain the estimates

$$\|u_{sc}\|_{W^1_{2,-\delta}(R^n)} \leq \frac{C}{k}, \quad \delta > \frac{1}{2}, \quad k > 1.$$

But in the nonlinear case we need to prove the solvability in the different type of spaces, namely in the Sobolev space $W^1_\infty(R^n)$ (due to nonlinearity).

Theorem (Direct problem)

Suppose that $\vec{W}(\cdot,1), V(\cdot,1), \beta_W, \beta_V$ *belong to* $L^p_{loc}(R^n), n = 1,2,3,$ $\max\{1; \frac{n}{2}\} < p \leq \infty$, *and have special behaviour at the infinity as* $O\left(\frac{1}{|x|^\mu}\right)$, $\mu > n$. *Then for any* $\rho > 0$ *there exists* $k_0 > 0$ *such that for all* $k \geq k_0$ *the Lippmann-Schwinger equation w.r.t.* u_{sc} *has a unique solution in the closed ball* $\overline{B}_\rho(0) = \{f \in W^1_\infty(R^n) : \|f\|_{W^1_\infty} \leq \rho\}$ *and*

$$\|u_{sc}\|_{L^\infty(R^n)} \leq \frac{C}{k^{\frac{5-n}{2}}}, \quad \|\nabla u_{sc}\|_{L^\infty(R^n)} \leq \frac{C}{k^{\frac{3-n}{2}}}$$

for all $k \geq k_0$. *For* $n = 1$ *we may consider these functions just from* $L^1(R)$.

In addition to the solvability result, the following asymptotic representation for these solutions as $|x| \to \infty$ holds

$$u(x,k,\theta) = u_0 + C_n \frac{k^{\frac{n-7}{2}} e^{ik|x|}}{|x|^{\frac{n-1}{2}}} A(k,\theta',\theta) + o\left(\frac{1}{|x|^{\frac{n-1}{2}}}\right).$$

Here the function $A(k,\theta,\theta')$ is called the scattering amplitude and is defined by

$$A(k,\theta',\theta) = \int\limits_{R^n} e^{-ik(\theta',y)} \left(\vec{W}(y,|u|)\nabla + V(y,|u|)\right) u(y,k,\theta)\, dy,$$

where $\theta' = \frac{x}{|x|}$ is the angle of observation. In $1D$ case $\theta, \theta' = \pm 1$.

The first result which concerns the inverse problems is the analogue of famous Saito's formula which was proved originally by Y. Saito for linear Schrödinger operator with smooth potentials.

Theorem (Saito's formula)

Suppose that $\vec{W}(\cdot,1), V(\cdot,1), \beta_W, \beta_V$ belong to $L^p_{loc}(R^n), n = 2, 3$ $n < p \leq \infty$, and have special behaviour at the infinity mentioned above. In addition we assume that $\nabla\vec{W}(\cdot,1) \in L^p_{loc}(R^n), n < p \leq \infty$ and has special behaviour at the infinity mentioned above. Then the limit

$$\lim_{k\to+\infty} k^{n-1} \int\limits_{S^{n-1}} \int\limits_{S^{n-1}} e^{-ik(\theta-\theta',x)} A(k,\theta',\theta)\, d\theta\, d\theta' =$$

$$= \frac{(2\pi)^n}{\pi} \int\limits_{R^n} \frac{V(y,1) - \frac{1}{2}\nabla\vec{W}(y,1)}{|x-y|^{n-1}}\, dy$$

holds uniformly for $n = 2$ and in the sense of distributions for $n = 3$.

It is also valid in $1D$ case under special interpretation of the left-hand side.

In $1D$ case uniformly in $x \in R$ we have that

$$\lim_{k\to+\infty}\Big[e^{-2ikx}\int_R e^{iky}\left(W(y,|u_+|)u_+' + V(y,|u_+|)u_+\right)\,dy+$$

$$e^{2ikx}\int_R e^{-iky}(W(y,|u_-|)u_-' + V(y,|u_-|)u_-)\,dy+$$

$$\int_R e^{-iky}(W(y,|u_+|)u_+' + V(y,|u_+|)u_+)\,dy+$$

$$\int_R e^{iky}(W(y,|u_-|)u_-' + V(y,|u_-|)u_-)\,dy\Big] = 2\int_R (V(y,1) - \frac{1}{2}W'(y,1))\,dy,$$

where $u_\pm$ are the scattering solutions of $H_4 u = k^4 u$ that behave as $u_\pm(y,k) \approx e^{\pm iky}$ with $u_\pm'(y,k) \approx \pm ike^{\pm iky}$ when $k \to +\infty$.

The significance of Saito's formula for inverse scattering problems is apparent from its corollaries.

Corollary (Uniqueness)

Let $\vec{W}_1, V_1$ and $\vec{W}_2, V_2$ be as in Saito's formula. If the corresponding scattering amplitudes $A_1(k, \theta', \theta)$ and $A_2(k, \theta', \theta)$ coincide for some sequence $k_j \to +\infty$ and for all angles θ, θ', then $V_1(\cdot, 1) - \frac{1}{2}\nabla \vec{W}_1(\cdot, 1) =$ $= V_2(\cdot, 1) - \frac{1}{2}\nabla \vec{W}_2(\cdot, 1)$ in the sense of tempered distributions (a.e. in R^n).

Corollary (representation formula)

Under the same assumptions as in Saito's formula

$$V(x,1) - \frac{1}{2}\nabla \vec{W}(x,1) =$$

$$= \frac{1}{2^{n+1}\pi^{2n-2}} \lim_{k \to +\infty} k^n \int_{S^{n-1}} \int_{S^{n-1}} e^{-ik(\theta-\theta', x)} |\theta - \theta'| A(k, \theta', \theta)\, d\theta\, d\theta'$$

in the sense of tempered distributions.

It is rather remarkable that the latter representation formula holds in $1D$ case also. Namely,

$$V(x,1) - \frac{1}{2}\nabla \vec{W}(x,1) =$$

$$= \frac{1}{2}\lim_{k\to+\infty} k\left(e^{-2ikx}A(k,-\theta,\theta) + e^{2ikx}A(k,\theta,-\theta)\right)$$

in the sense of tempered distributions. Here the scattering amplitudes $A(k,-\theta,\theta)$ and $A(k,\theta,-\theta)$ are interpreted as before in $1D$ case (see slides above), where $\theta = \pm 1$.

In the linear case (i.e., in the case when no dependence on $|u|$ in V and $\vec{W}$) we can use different type of data for inverse problem. Namely, the estimates for the operator $\left(\Delta^2 - k^4 - i0\right)^{-1}$ can be applied to the operator $\left(H_4 - k^4 - i0\right)^{-1}$. This operator exists as the limit $\lim\limits_{\epsilon \to +0} \left(H_4 - k^4 - i\epsilon\right)^{-1}$ in the operator topology from $L^2_\delta(R^n)$ to $W^1_{2,-\delta}(R^n)$ (with the same δ as above) with the norm estimate

$$\| \left(H_4 - k^4 - i0\right)^{-1} f \|_{W^1_{2,-\delta}} \leq \frac{C}{k^2} \|f\|_{L^2_\delta}.$$

Moreover, this operator is an integral operator with the kernel $G(x, y, k)$ which satisfies the integral equation

$$G(x, y, k) = G_k^+(|x-y|) - \int\limits_{R^n} G_k^+(|x-z|) \left(\vec{W}(z)\nabla_z + V(z)\right) G(z, y, k)\, dz.$$

The solvability of this equation (in the weighted Sobolev spaces) for k big enough follows from the norm estimate of the resolvent for H_4. But even more is true, the following solvability result holds in "L^∞" space.

We consider now $n = 3$ (for simplicity)

Proposition

Under the same assumption for $\vec{W}$ and V as in Saito's formula there is a constant $k_0 > 1$ such that the function $G(x, y, k)$ can be defined by the series of iterations

$$G(x, y, k) = \sum_{j=0}^{\infty} G^{(j)}(x, y, k), \quad G^{(0)} = G_k^+$$

which solves this integral equation uniquely, when $k \geq k_0$, and

$$|G(x, y, k) - G_k^+(x, y, k)| \leq \frac{c_0}{4\pi^2 k^3 |x - y|},$$

$$|\nabla G(x, y, k) - \nabla G_k^+(x, y, k)| \leq \frac{c_0}{2\pi^2 k^2 |x - y|}.$$

The knowledge of $G(x, y, k)$ for large values of k, x, y leads to the result.

Theorem

Let $\xi \in R^3$ be arbitrary and fixed. Assume that $\vec{W}$ and V are as above. Then for $\xi = -k\left(\frac{x}{|x|} + \frac{y}{|y|}\right)$

$$\mathcal{F}^{-1}\left(V - \frac{1}{2}\nabla\vec{W}\right)(\xi) =$$

$$= 32\sqrt{2\pi}\lim_{x,y,k\to\infty} k^4|x||y|e^{-ik(|x|+|y|)}\left(G_k^+(|x-y|) - G(x,y,k)\right),$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform in R^3.

As an immediate corollary we have the uniqueness result for this inverse scattering problem. If $G_1(x, y, k)$ and $G_2(x, y, k)$ are two kernels which correspond to two pairs $\vec{W}_1, V_1$ and $\vec{W}_2, V_2$ and if $G_1(x, y, k)$ and $G_2(x, y, k)$ coincide for all x, y, k big enough then $V_1 - \frac{1}{2}\nabla\vec{W}_1 =$
$= V_2 - \frac{1}{2}\nabla\vec{W}_2$ a.e. in R^3.

In the inverse scattering theory there is very effective and very applicable approximate method which is called the Born approximation. We consider this method for the backscattering problem, i.e., for the problem when the angle of observation θ' is equal to minus angle of incident wave $\theta, \theta' = -\theta$. It can be proved in this case that

$$A(k, -\theta, \theta) \approx -\frac{1}{2}\mathcal{F}(\nabla \vec{W})(2k\theta) + \mathcal{F}(V)(2k\theta), \quad k \to +\infty.$$

This fact justifies the following definition

Definition

The inverse backscattering Born approximation $V_B^b(x)$ for the operator H_4 is defined as

$$V_B^b(x) := \mathcal{F}^{-1}\left(A\left(\frac{k}{2}, -\theta, \theta\right)\right)(x) =$$

$$= \frac{1}{(2\pi)^n} \int\limits_{R_+ \times S^{n-1}} k^{n-1} e^{-ik(x,\theta)} A\left(\frac{k}{2}, -\theta, \theta\right) d\theta\, dk.$$

In the absence of uniqueness the following result is valid.

Theorem (Reconstruction of singularities. Linear case)

Suppose that $\vec{W}$ belongs to the weighted Sobolev space $W^1_{p,\delta}(R^n)$ and V belongs to $L^p_\delta(R^n)$ where $n < p \leq \infty$ and $\delta > \frac{n}{p'}, n = 2, 3$. Then the difference

$$V^b_B(x) - \left(V(x) - \frac{1}{2}\nabla\vec{W}(x) \right) \in W^t_2(R^n) \quad (mod C^\infty(R^n))$$

for any $t < \frac{6-n}{2}$.

This theorem means that using the inverse backscattering Born approximation we can uniquely determine all main singularities in $3D$ case (all singularities and jumps in $2D$ case) of the combination $V(x) - \frac{1}{2}\nabla\vec{W}(x)$ since

$$W_2^{\frac{6-n}{2}}(R^n) \subset W_p^{\frac{n}{p}+3-n}(R^n), \quad n < p \leq \infty, \quad n = 2, 3.$$

Moreover, any smooth $3D$ bounded domain (and any $2D$ domain) ($p = \infty$) can be uniquely determined using this Born approximation.

In order to apply the inverse backscattering Born approximation in the quasi-linear case we assume in addition that (we consider $n = 2$ only)

$$\vec{W}(x, 1+s) = \vec{W}(x, 1) + s\vec{W}^*(x, 1) + O(s^2)\vec{W}^{**}(x, s^*)$$
$$V(x, 1+s) = V(x, 1) + sV^*(x, 1) + O(s^2)V^{**}(x, s_0^*),$$

where $|s^*|, \ |s_0^*| < |s|$. We assume that $\vec{W}(\cdot, 1), \vec{W}^*(\cdot, 1) \in W^1_{p,loc}(R^2)$ and $V(\cdot, 1), V^*(\cdot, 1) \in L^p_{loc}(R^2)$, with some $\frac{4}{3} < p \leq \infty$, and for all $f \in \{\vec{W}(\cdot, 1)\vec{W}^*(\cdot, 1), \nabla \cdot \vec{W}(\cdot, 1), \nabla \cdot \vec{W}^*(\cdot, 1), V(\cdot, 1), V^*(\cdot, 1)\}$ the following decay properties are satisfied

$$|f(x)| \leq \frac{C}{|x|^\mu}, \quad |x| > R,$$

where $R > 0$ is arbitrary large and $\mu > 2$. Functions $\vec{W}^{**}$ and V^{**} are assumed to be in $L^q_{loc}(R^2)$, with some $\frac{4}{3} < q \leq \infty$ and $q = 1$ respectively, and they satisfy the decay property from above, uniformly in s^* and s_0^*, respectively. All physically relevant nonlinearities satisfy these conditions.

The main result here is the following theorem

Theorem (Reconstruction of singularities. Nonlinear case)

Let functions $\vec{W}$ and V be as above with $2 \leq p \leq \infty$. Then the difference

$$V_B^b(x) - \left(V(x) - \frac{1}{2}\nabla\vec{W}(x) \right) \in H^t(R^2) \pmod{C^\infty(R^2)}$$

for any $t < 1$.

Corollary

If $\vec{W}$ and V are as in the Main theorem then the jumps of $V(x) - \frac{1}{2}\nabla\vec{W}(x)$ over smooth curves are uniquely determined by the backscattering data and can be recovered from $V_B^b(x)$.

The scattering solutions in $1D$ case admit the following asymptotical representation as $x \to \pm\infty$ ($k \geq k_0 > 0$ fixed)

$$u(x,k) = a(k)e^{ikx} + o(1), \quad x \to +\infty$$

$$u(x,k) = e^{ikx} + b(k)e^{-ikx} + o(1), \quad x \to -\infty,$$

where $a(k)$ and $b(k)$ are called transmission and reflection coefficients, respectively, and defined as

$$a(k) = 1 - \frac{i}{4k^3} \int_{-\infty}^{\infty} e^{-iky} (W(y,|u|)u'(y) + V(y,|u|)u(y))\, dy$$

$$b(k) = -\frac{i}{4k^3} \int_{-\infty}^{\infty} e^{iky} (W(y,|u|)u'(y) + V(y,|u|)u(y))\, dy$$

We are interested further only the reflection coefficient.

The asymptotical representations for $u(x,k)$ for large x can be considered as the analogue of the one-dimensional Sommerfeld radiation conditions for this operator of order 4. But what is more important, the asymptotic of $u(x,k)$ (or of b) for large k justifies the following definition which plays the crucial role in the inverse scattering problem.

Definition

The inverse scattering Born approximation $V_B(x)$ of the potential β is defined by

$$V_B(x) := \mathcal{F}^{-1}\left(\frac{i}{2\sqrt{2\pi}}k^3 b\left(\frac{k}{2}\right)\right),$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform on the line.

This equality must be considered in the sense of tempered distributions.

We are in the position now to formulate the main result in $1D$ case.

Theorem

If $V(x,1) \in L^1(R)$, $W(x,1) \in W_1^1(R)$ and $V(x,\cdot), W(x,\cdot)$ are as before, then the inverse scattering Born approximation V_B admits the representation

$$V_B(x) = \Re(\beta)(x) + \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \frac{\Im(\beta)(y)}{x-y}\, dy \quad (mod C_0(R))$$

where $\beta(x) = V(x,1) - \frac{1}{2}W'(x,1)$.

Corollary

If V, W and β are as in Theorem and in addition are real-valued, then the difference

$$V_B(x) - \beta(x) \in C_0(R).$$

In order to obtain this result we were needed to calculate precisely the so-called "first" nonlinear term in the Born sequence and investigate its smoothness (using this precise form). Without it we are not able to obtain the required result. Indeed, let us substitute $u(x,k) = e^{ikx} + u_{sc}(x,k)$ into $b(k)$ then it can be easily seen that

$$V_B(x) = \beta(x) + V_{rest}(x) \quad (mod \quad C^{\infty}(R)).$$

In order to estimate the smoothness of V_{rest} we first remark that

$$|k^3 b_{sc}(k)| \leq \frac{C}{|k|}, \quad |k| \geq \sqrt{2c_0},$$

where b_{sc} is a part of $b(k)$ which corresponds to u_{sc}. This implies that

$$\|V_{rest}\|^2_{H^t(R)} \leq C \int\limits_{\sqrt{2c_0}}^{\infty} \frac{(1+k^2)^t}{k^2}\, dk < \infty$$

for any $t < \frac{1}{2}$. Using then Sobolev imbedding theorem we obtain that for arbitrary positive ϵ (small enough)

$$V_{rest}(x) \in W_p^{\frac{1}{p}-\epsilon}(R), \quad 2 \leq p < \infty.$$

There is an explanation why we need to investigate the first nonlinear term in the Born series (in addition). The smoothness of V_{rest} shows that we are not able to consider $p = \infty$. It means that if we do not use the first nonlinear term then it is possible to reconstruct any singularity from L^p with $p < \infty$ but not from L^∞ (i.e., jumps).
There is one more corollary.

Corollary

If V, W and β are as in Theorem and $\Im(\beta) \in H^r(R)$ for $r > \frac{1}{2}$, then the difference

$$V_B(x) - \Re(\beta)(x) \in C_0(R).$$

Remark

According to these corollaries, if W is smooth enough (or $\Im(\beta)$ in the complex case), then we can recover any local L^p-singularities and any jumps of the unknown potential $V(x, 1)$ using the Born approximation. However, we cannot distinguish the singularities and jumps of V and W' if both are discontinuous.

Let us consider now linear one-dimensional 4th order equation in the self-adjoint form

$$u^{(4)}(x) + 2iW(x)u'(x) + iW'(x)u(x) + V(x)u(x) = k^4 u(x), \quad x \in R,$$

where W, V are real-valued functions. In that case we can prove the following result.

Theorem

Assume that the potentials $W(x)$ and $V(x)$ belong to the Sobolev space $W_1^1(R)$ and the Lebesgue space $L^1(R)$, respectively. Then

$$V_B(x) - V(x) \in C_0(R),$$

i.e., this difference is continuous function everywhere on the line and therefore, all jumps and singularities of V are uniquely determined by the Born approximation V_B.

Summary

The theorems proved show that all singularities (in some cases also all jumps) of the unknown function $\beta = V - \frac{1}{2}\nabla\vec{W}$ (the potential V in the self-adjoint case) can be obtained exactly by the inverse scattering Born approximation. In particular in $1D$ case (this is the most practical application of this result), we can prove that for the function $V(x)$ being the characteristic function of an interval on the line, this interval is uniquely determined by the scattering data. Moreover, in the class of such potentials we have even uniqueness result of the inverse scattering problem.

Thanks very much for your attention !