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On direct and inverse Kolmogorov equations for purely jump-like Markov processes and their generalizations [†]

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In the work **“On analytical methods in probability theory”** (1931), A. N. Kolmogorov, starting from the relations called **Kolmogorov–Chapman equations**, derived for transition probabilities of inhomogeneous stochastically defined systems, or, as is now commonly said, for inhomogeneous Markov random processes (in an expanded meaning),

reverse and **direct**

equations in the following three cases:

(A) systems with a **finite** number of states;

(B) systems with **countable** number of states;

(C) diffusion-type systems with a **continuous** set of states.

The report, which is largely of a review nature, considers the cases (A), (B) and the purely jump case for a Markov process with a Borel state space.

In “Analytical methods” direct and inverse equations were derived under the assumption of **differentiability** of transition probabilities with respect to temporary variables.

Later, A. N. Kolmogorov especially emphasized that the question of differentiability is far from simple: this property of transition probabilities significantly depends on the behavior of the paths of the system under consideration, for example, on such properties as “explosiveness”, etc. .

In the case of (A) (finite number of states), W. Doeblin showed that the differentiability of transition functions follows from the (constantly used) assumption of stochastic continuity of the Markov process under consideration. Thus, in the case of a finite set of states, the question of the validity of inverse and direct equations was completely resolved.

Case (B) (countable case) is much more complicated. In the article “On the question of differentiability of transition probabilities..." (1951) A. N. Kolmogorov noted the arising here features, the study of which contributed to the development of many important concepts, such as, as a strictly Markov property.

We do not touch here on the case (C), which is also actively studied, especially in the framework of Ito's stochastic differential equations for diffusion processes.

In 1940, W. Feller considered a generalization of the cases (A) and (B): he studied the question of inverse and direct equations for (purely) jump Markov processes with continuous time taking values in Polish spaces .

In a series of our works devoted to direct and inverse Kolmogorov equations for general (purely) jump Markov processes and their applications in optimal control, these equations are considered under weaker assumptions than those made by W. Feller.

The report will present results on inverse and direct equations in the cases of (A) and (B); we will also present, following our works mentioned above, very general conditions for the validity of inverse and direct equations for (purely) jump Markov processes with values in standard Borel space.

2. Case (A) (finite set of states)

Let a **homogeneous** Markov process $X = (X_t)_{t \geq 0}$ with a finite set of states $\mathcal{N} = \{1, 2, \ldots, n\}$, whose paths are continuous on the right for all $t > 0$ and are piecewise constant (the so-called **pure** case), be given on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

2.1. Consider transition probabilities

$$p_{ij}(t) = \mathbf{P}(X_t = j \mid X_0 = i), \quad \text{где } i, j \in \mathcal{N},\ t \geq 0. \tag{1}$$

They have the following properties:

$$p_{ij}(t) \geq 0, \quad p_{ij}(0) = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j; \end{cases} \tag{2}$$

$$\sum_{j \in \mathcal{N}} p_{ij}(t) = 1 \quad \text{for all } i \in \mathcal{N}; \tag{3}$$

$$p_{ij}(t+s) = \sum_{k \in \mathcal{N}} p_{ik}(t) p_{kj}(s), \quad t, s \geq 0. \tag{4}$$

The Markov (semigroup) property (4) is obviously a variant of the Kolmogorov–Chapman equation.

Everywhere below we assume that the process $X = (X_t)_{t \geq 0}$ is **stochastically continuous**, i.e. for any $\varepsilon > 0$

$$\mathbf{P}(|X_{t+h} - X_t| > \varepsilon) \to 0, \qquad t \geq 0, \tag{5}$$

when $h \to 0$ (for $t = 0$ we assume here and below that $h \downarrow 0$).

Condition (5) is equivalent to the fact that

$$\lim_{t \downarrow 0} p_{ij}(t) = \delta_{ij}, \qquad i, j \in \mathcal{N}. \tag{6}$$

The following results can be considered well known.

Proposition 1. (a) There exist finite limits

$$q_{ij} = \lim_{h \downarrow 0} \frac{p_{ij}(h) - \delta_{ij}}{h}, \tag{7}$$

in other words, there exist finite limits

$$q_{ij} = \lim_{h \downarrow 0} \frac{p_{ij}(h)}{h} \; (\geq 0) \quad \text{for } i \neq j, \tag{8}$$

$$q_{ii} = \lim_{h \downarrow 0} \frac{p_{ii}(h) - 1}{h} \; (\leq 0) \quad \text{for all } i \in \mathcal{N}. \tag{9}$$

Moreover, for all $i \in \mathcal{N}$

$$\sum_{j \neq i} q_{ij} = -q_{ii}, \quad \text{i. e.} \quad \sum_{j \in \mathcal{N}} q_{ij} = 0.$$

(b) For all $i, j \in \mathcal{N}$ there exist derivatives $dp_{ij}(t)/dt$.

(c) Transition probabilities $p_{ij}(t)$ satisfy (with the initial condition $p_{ij}(0) = \delta_{ij}$) **direct** equation

$$\frac{dp_{ij}(t)}{dt} = \sum_{k \in \mathcal{N}} p_{ik}(t) q_{kj}, \tag{10}$$

i. e. (with $q_i = -q_{ii},\ i \in \mathcal{N}$) the equation

$$\frac{dp_{ij}(t)}{dt} = -p_{ij}(t) q_j + \sum_{k \neq j} p_{ik}(t) q_{kj}, \tag{11}$$

and **inverse** equation

$$\frac{dp_{ij}(t)}{dt} = \sum_{k \in \mathcal{N}} q_{ik} p_{kj}(t), \tag{12}$$

i. e. the equation

$$\frac{dp_{ij}(t)}{dt} = -q_i p_{ij}(t) + \sum_{k \neq j} q_{ik} p_{kj}(t). \tag{13}$$

Note that (under the assumption of **differentiability** $p_{ij}(t)$) the equations (10), (12) immediately follow from the Markov property (4), i. e. from the Kolmogorov–Chapman equation.

Indeed, to obtain a direct equation (10), we must use the fact that

$$\frac{p_{ij}(t+h) - p_{ij}(t)}{h} = \sum_{k \in \mathcal{N}} p_{ik}(t) \, \frac{p_{kj}(h) - \delta_{kj}}{h}. \tag{14}$$

And to obtain the inverse equation (12) one should use the fact that

$$\frac{p_{ij}(t+h) - p_{ij}(t)}{h} = \sum_{k \in \mathcal{N}} \frac{p_{ik}(h) - \delta_{ik}}{h} \, p_{kj}(t). \tag{15}$$

From the existence of limits (7) and (9) and **finiteness** of the number of states, we obtain statements about the existence and finiteness of the derivatives $dp_{ij}(t)/dt$.

In matrix form, the equations (10), (12) can be written as follows:

$$P'(t) = P(t)Q \quad \text{and} \quad P'(t) = QP(t),$$

where $P(t) = \|p_{ij}(t)\|$, $Q = \|q_{ij}\|$.

2.2. The local characteristics q_{ij}, $i, j \in \mathcal{N}$ can be given a useful interpretation if we use the concept of an **embedded** chain. This is the name given to a Markov chain $Z = (Z_t)_{t \geq 0}$, where $Z_0 = X_0$ and $Z_n = X_{\tau_n}$, $n \geq 1$, and τ_n is the moment of the n^{th} jump of the process $X = (X_t)_{t \geq 0}$.

Then

$$\mathbf{P}(\tau_1 > t \,|\, X_0 = i) = e^{-q_i t} \tag{16}$$

and, due to the homogeneity of the process X,

$$\mathbf{P}(\tau_n - \tau_{n-1} > t \,|\, X_{\tau_{n-1}} = i) = e^{-q_i t} \tag{17}$$

for any $n \geq 2$.

To prove (16), remark that

$$\mathbf{P}(\tau_1 > t \mid X_0 = i) = \lim_{n\to\infty} \mathbf{P}(X_{t_{nk}} = i,\, k = 0, 1, \ldots, n \mid X_0 = i),$$

where $0 = t_{n0} < t_{n1} < \cdots < t_{nn} = t$ such that $\max_{1\le k\le n}|t_{nk} - t_{n(k-1)}| \to 0$ as $n \to \infty$. Then

$$\begin{aligned}\mathbf{P}(\tau_1 > t \mid X_0 = i) &= \lim_{n\to\infty} \prod_{k=1}^{n} p_{ii}(t_{nk} - t_{n(k-1)}) \\ &= \exp\Big\{\lim_{n\to\infty} \sum_{k=1}^{n} \ln[p_{ii}(t_{nk} - t_{n(k-1)})]\Big\} = e^{-q_i t},\end{aligned}$$

since for large n

$$\begin{aligned}\ln[p_{ii}(t_{nk} - t_{n(k-1)})] &\sim p_{ii}(t_{nk} - t_{n(k-1)}) \\ &\sim q_{ii} \cdot (t_{nk} - t_{n(k-1)}) = -q_i \cdot (t_{nk} - t_{n(k-1)}),\end{aligned}$$

which leads to the formula(16).

It also can be shown that for $j \neq i$

$$\mathbf{P}(X_{\tau_1} = j \,|\, X_0 = i) = \frac{q_{ij}}{q_i} \tag{18}$$

and, again due to homogeneity, for $n \geq 2$

$$\mathbf{P}(X_{\tau_n} = j \,|\, X_{\tau_{n-1}} = i) = \frac{q_{ij}}{q_i}, \qquad i \neq j. \tag{19}$$

2.3. Until now, we assumed that a homogeneous Markov chain $X = (X_t)_{t \geq 0}$ with continuous time is defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Based on the transition probabilities $p_{ij}(t) = \mathbf{P}(X_t = j \,|\, X_0 = i)$, local characteristics q_{ij}, $i, j \in \mathcal{N} = \{1, 2, \ldots, n\}$ were calculated.

But one can reason, in a certain sense, in the opposite direction, namely, consider given finite quantities q_{ij}, $i, j \in \mathcal{N}$, such that $q_{ij} \geq 0$ for $i \neq j$ and $q_{ii} \leq 0$, additionally assuming that $\sum_{j \neq i} q_{ij} = -q_{ii}$ for all $i \in \mathcal{N}$.

Then, using these characteristics, it is possible to construct, guided by the formulas (16)–(19), a corresponding embedded Markov chain, the probability distribution of which is such that its transition probabilities satisfy the direct and inverse equations (10) and (12).

2.4. Above we considered the case of a **homogeneous** Markov process $X = (X_t)_{t \geq 0}$. If this process is inhomogeneous, then its transition probabilities

$$p_{ij}(s,t) = \mathbf{P}(X_t = j \mid X_s = i), \qquad s \leq t,$$

will already depend on two variables s and t. Then inverse equations are equations depending on s, and direct equations are equations depending on t.

In order for these equations to hold, certain assumptions must be made.

We will assume that

$$p_{ij}(s,t) = \begin{cases} 1 + q_{ii}(t)(t-s) + o(t-s), & i = j, \\ q_{ij}(t)(t-s) + o(t-s), & i \neq j, \end{cases} \tag{20}$$

and the functions $q_{ij}(t)$ are continuous in t for all $i, j \in \mathcal{N}$.

These conditions are sufficient for the following direct and inverse equations to take place:

$$\frac{\partial p_{ij}(s,t)}{\partial t} = -p_{ij}(s,t)q_j(t) + \sum_{k \neq j} p_{ik}(s,t)q_{kj}(t), \tag{21}$$

$$\frac{\partial p_{ij}(s,t)}{\partial s} = q_i(s)p_{ij}(s,t) - \sum_{k \neq j} q_{ik}(s)p_{kj}(s,t), \tag{22}$$

where $q_j(t) = -q_{jj}(t)$.

Under the stated conditions, the formulas (21) and (22) are derived by considering the increments

$$\frac{1}{h}\left[p_{ij}(s,t+h)-p_{ij}(s,t)\right] \quad \text{и} \quad \frac{1}{h}\left[p_{ij}(s+h,t)-p_{ij}(s,t)\right]$$

quantities (14) and (15)), to which the Kolmogorov–Chapman equation is applied, and then the passage to the limit is made as $h \to 0$. The resulting (partial) derivatives are right derivatives, which are continuous and therefore coincide with the derivatives $\partial p_{ij}(s,t)/\partial t$ and $\partial p_{ij}(s,t)/\partial s$.

3. Case (B) (countable set of states)

The case of a Markov process $X = (X_t)_{t \geq 0}$ with a countable set of states, as already noted, turned out to be more complex, and this is due to the fact that effects arise here that are absent in the case of a finite set of states (case (A)). Such effects include, for example, the appearance of **instantaneous** states and the occurrence of "explosions" at a random finite moment in time. As a consequence, situations are possible when inverse equations exist, but direct ones do not.

Let us present the main results here, starting with a **homogeneous** (stochastically continuous) process.

Proposition 2. (a) There exist the limits

$$q_{ij} = \lim_{h \downarrow 0} \frac{p_{ij}(h) - \delta_{ij}}{h}, \tag{23}$$

where $p_{ij}(h) = \mathbf{P}(X_h = j \mid X_0 = i)$; moreover, if $i \neq j$, then these limits are finite. However, there are possible cases when $q_{ii} = -\infty$ for some (and even all) $i \in \mathbf{N} = \{1, 2, \ldots\}$.

There is always inequality

$$\sum_{j \neq i} q_{ij} \leq -q_{ii}, \tag{24}$$

and, even for finite q_{ii}, cases are possible when

$$\sum_{j \neq i} q_{ij} < -q_{ii}. \tag{25}$$

(b) If all states $i \in \mathbf{N}$ are **non-instantaneous** (i.e. $q_i = -q_{ii} < \infty$), then for the existence of derivatives $dp_{ij}(t)/dt$, $i, j \in \mathbf{N}$, and execution of the inverse system

$$\frac{dp_{ij}(t)}{dt} = \sum_{k \in \mathbf{N}} q_{ik} p_{kj}(t), \tag{26}$$

or

$$\frac{dp_{ij}(t)}{dt} = -q_i p_{ij}(t) + \sum_{k \neq j} q_{ik} p_{kj}(t), \tag{27}$$

it is necessary and sufficient that all states $i \in \mathbf{N}$ be **regular**, i.e. that the following equalities hold:

$$\sum_{j \neq i} q_{ij} = -q_{ii} \; (= q_i < \infty). \tag{28}$$

(c) Let all states $i \in \mathbf{N}$ be non-instantaneous and regular, i. e. $q_i < \infty$ and $\sum_{j \neq i} q_{ij} = q_i$, $i \in \mathbf{N}$. There are examples when direct equations

$$\frac{dp_{ij}(t)}{dt} = \sum_{k \in \mathbf{N}} p_{ik}(t) q_{kj}, \quad \text{or} \quad \frac{dp_{ij}(t)}{dt} = -p_{ij}(t) q_j + \sum_{k \neq j} p_{ik}(t) q_{kj},$$

do not hold. If, however, for a given i^* and all $t > 0$

$$\sum_{k \in \mathbf{N}} p_{i^* k}(t) q_{kk} > -\infty, \tag{29}$$

then for i^* and all $j \in \mathbf{N}$ the following direct equations are valid:

$$\frac{dp_{i^* j}(t)}{dt} = \sum_{k \in \mathbf{N}} p_{i^* k}(t) q_{kj}, \quad \text{or} \quad \frac{dp_{i^* j}(t)}{dt} = -p_{i^* j}(t) q_j + \sum_{k \neq j} p_{i^* k}(t) q_{kj}.$$

Generally speaking, it does not follow from anywhere that, say, a system of inverse equations has a unique, and also probabilistic, solution. We therefore investigate the question of the uniqueness of the solution $f = (f_{ij}(t))_{t \geq 0}$ of the (inverse) system

$$\frac{df_{ij}(t)}{dt} = \sum_{k \in \mathbf{N}} q_{ik} f_{kj}(t), \qquad f_{ij}(0) = \delta_{ij}, \tag{30}$$

where $q_{ij} \geq 0$ if $i \neq j$, $q_{ii} \leq 0$ and $\sum_{k \in \mathbf{N}} q_{ik} = 0$ for each $i \in \mathbf{N}$. We will also assume that all quantities q_{ij} are finite.

Remark 1. The results presented above for homogeneous jump Markov processes with a countable set of states are a special case of the results for nonhomogeneous jump Markov processes with a Borel set of states. The corresponding more general results are presented in Section 4: for inverse equations in the theorem 1, and for direct equations in the theorem 2.

The system (30) can be written in the form

$$\frac{df_{ij}(t)}{dt} + q_i f_{ij}(t) = \sum_{k \neq i} q_{ik} f_{kj}(t), \qquad f_{ij}(0) = \delta_{ij}, \tag{31}$$

which, in turn, is rewritten as follows:

$$\frac{d}{ds}[f_{ij}(s)e^{q_i s}] = e^{q_i s} \sum_{k \neq i} q_{ik} f_{kj}(s), \qquad f_{ij}(0) = \delta_{ij}.$$

After integration over s in the limits from 0 to t, we obtain that differential equations (31) are equivalent to integral equations

$$f_{ij}(t) = \delta_{ij} e^{-q_i t} + \int_0^t e^{-q_i(t-s)} \sum_{k \neq i} q_{ik} f_{kj}(s)\, ds. \tag{32}$$

To find the solution $f = (f_{ij}(t))_{t \geq 0}$ of this equation, the method of successive approximations can be used. According to this method we put

$$f_{ij}^{(0)}(t) = \delta_{ij} e^{-q_i t},$$
$$f_{ij}^{(n)}(t) = \delta_{ij} e^{-q_i t} + \int_0^t e^{-q_i(t-s)} \sum_{k \neq i} q_{ik} f_{kj}^{(n-1)}(s)\, ds, \qquad n \geq 1. \tag{33}$$

Then, since $0 \leq f_{ij}^{(0)}(t) \leq f_{ij}^{(1)}(t) \leq \cdots$, we see that there is a limit

$$\bar{f}_{ij}(t) = \lim_{n \to \infty} f_{ij}^{(n)}(t). \tag{34}$$

Moreover, for each $n \geq 0$ and any $i \in \mathbf{N}$

$$\sum_j f_{ij}^{(n)}(t) \leq 1. \tag{35}$$

Indeed, let $g_i^{(n)}(t) = \sum_j f_{ij}^{(n)}(t)$. Then it is clear that

$$g_i^{(0)}(t) = \sum_j f_{ij}^{(0)}(t) = e^{-q_i t} \leq 1,$$

$$g_i^{(n)}(t) = \sum_j f_{ij}^{(n)}(t) = e^{-q_i t} + \int_0^t e^{-q_i(t-s)} \sum_{k \neq i} q_{ik} g_k^{(n-1)}(s)\, ds.$$

Therefore, if $g_i^{(n-1)}(t) \leq 1$, then, since it is assumed that $\sum_{k \neq i} q_{ik} = -q_{ii}$, we have

$$g_i^{(n)}(t) \leq e^{-q_i t} + \int_0^t e^{-q_i(t-s)} \sum_{k \neq i} q_{ik}\, ds = e^{-q_i t} + q_i \int_0^t e^{-q_i(t-s)}\, ds = 1.$$

Hence, since $g_i^{(0)}(t) \leq 1$, we obtain the required inequality (35), and due to (34) we find that for each $i \in \mathbf{N}$ the following inequality holds:

$$\sum_j \bar{f}_{ij}(t) \leq 1. \qquad (36)$$

The solution $\bar{f}_{ij}(t)$ found in this way is **the smallest** among all other non-negative solutions $\tilde{f}_{ij}(t)$ of the integral equation (31).

Indeed, from (31) it is clear that $\tilde{f}_{ij}(t) \geq \delta_{ij} e^{-q_i t} = f_{ij}^{(0)}(t)$, and by induction it is easy to establish that $\tilde{f}_{ij}(t) \geq f_{ij}^{(n)}(t)$, and therefore $\tilde{f}_{ij}(t) \geq \bar{f}_{ij}(t)$.

Let us now assume that the found solution $\bar{f}_{ij}(t)$ is probabilistic, i.e. $\sum_j \bar{f}_{ij}(t) = 1$ for all $i \in \mathbf{N}$. Then it is the **unique** probabilistic solution of the equation (32).

Indeed, let $\overline{\mathcal{P}}$ denote the class of all probabilistic solutions of the equation (32), and let $\tilde{f}_{ij}(t)$ be another solution from $\overline{\mathcal{P}}$. Then

$$\sum_j [\tilde{f}_{ij}(t) - \bar{f}_{ij}(t)] = 0.$$

But $\tilde{f}_{ij}(t) \geq \bar{f}_{ij}(t)$. Therefore, this solution $\tilde{f}_{ij}(t)$ coincides with $\bar{f}_{ij}(t)$. Thus, the class $\overline{\mathcal{P}}$ consists of only one solution $\bar{f}_{ij}(t)$.

4. General pure jump Markov processes with piecewise constant, right continuous paths

As already noted, the transition from a countable set of states (case (B)) to the more general case of phase space was first made by W. Feller in 1940.

Further, following our works, we consider this general case under weaker restrictions on the so-called Q-functions, which are natural analogues of probability densities of exiting states (values q_i, $i \in \mathbf{N}$) and probability densities of transition (values q_{ij}, $i \neq j$).

We will now assume that the time interval on which Markov systems evolve is the set $[T_0, T_1)$, where $0 \le T_0 < T_1 \le \infty$.

As a phase space we will consider some standard Borel space $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$, defined, as is known, by the fact that $\mathbf{X}$ is a topological space with a metrizable topology transforming $\mathbf{X}$ into a Polish (i.e., complete separable metric) space, and $\mathcal{B}(\mathbf{X})$ is the sigma-algebra generated by open sets.

Let on the filtered probability space

$$(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P}),$$

a random Markov process $X = (X_t)$, $t \in [T_0, T_1)$, with phase space $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$, be defined:

$$\mathbf{P}(X_t \in B \,|\, \mathcal{F}_s) = \mathbf{P}(X_t \in B \,|\, X_s) \quad (\mathbf{P}\text{-п. н.}) \tag{37}$$

for all $B \in \mathcal{B}(\mathbf{X})$, $T_0 \le s \le t < T_1$.

According to the results of S. E. Kuznetsov (TVP, 1980), for the standard Borel space $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$ there is a (unique) transition function $P(s, x; t, B)$ such that

$$\mathbf{P}(X_t \in B \,|\, X_s) = P(s, X; t, B) \quad (\mathbf{P}\text{-п. н.}). \tag{38}$$

Recall that the function $P(s,x;t,B)$, where $s \le t$, $x \in \mathbf{X}$, $B \in \mathcal{B}(\mathbf{X})$, is called bluetransition function if (i) for all $s \le t$, $x \in \mathbf{X}$ the function $P(s,x;t,\cdot)$ is a measure on $(\mathbf{X},\mathcal{B}(\mathbf{X}))$ and $0 \le P(s,x;t,\cdot) \le 1$;

(ii) for all $B \in \mathcal{B}(\mathbf{X})$ the function $P(s,x;t,B)$ is Borel in s, x, t;

(iii) the function $P(s,x;t,B)$ satisfies the Kolmogorov–Chapman equation

$$P(s,x;t,B) = \int_{\mathbf{X}} P(u,y;t,B)\, P(s,x;u,dy), \qquad (39)$$

where $s < u < t$.

In the case when $P(u,y;t,\mathbf{X}) = 1$, the transition function is called **probabilistic**.

So, in the case of a standard Borel space, each Markov process is associated with a probabilistic transition function. In a certain sense, the opposite result is also true. Namely, from such a function and a given probability measure $\gamma = \gamma(dx)$, $x \in \mathbf{X}$, one can construct a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a probability process $X = (X_t)$, $t \in [T_0, T_1)$ on it, for which $\mathbf{P}(X_{T_0} \in dx) = \gamma(dx)$ and the transition probabilities are originally specified ones. This result follows from Kolmogorov's general results on the extension of a measure and the construction of the corresponding process in a coordinate manner (generally speaking, with "bad" paths).

Let us now turn to the already mentioned Q-functions.

Definition 1. The real function $q = q(t, x; B)$, where $t \in [T_0, T_1)$, $x \in \mathbf{X}$, $B \in \mathcal{B}(\mathbf{X})$, is called Q-function if the following two conditions are fulfilled:

(a) for fixed t, x this is a signed measure on $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$ such that $q(t, x; \mathbf{X}) \leq 0$ and

$$0 \leq q(t, x; B \setminus \{x\}) < \infty \quad \text{for all } B \in \mathcal{B}(\mathbf{X}); \tag{40}$$

(b) for all $B \in \mathcal{B}(\mathbf{X})$ the function $q(t, x; B)$ is measurable in t, x.

It is clear that in the case $\mathbf{X} = \mathbf{N}$

$$q(t, i; \{j\}) = q_{ij}(t), \quad q(t, i; \{i\}) = q_{ii}(t) = -q_i(t).$$

Let us put

$$q(t,x) = -q(t,x;\{x\}) \tag{41}$$

and

$$\overline{q}(x) = \sup_{t \in [T_0, T_1)} q(t,x). \tag{42}$$

Let us formulate a number of assumptions regarding the Q-function under which direct and inverse equations are considered.

Condition I (W. Feller). For all $n \geq 1$ there exist Borel subsets $B_n \in \mathcal{B}(\mathbf{X})$ such that $B_n \subseteq B_{n+1}$, $B_n \uparrow \mathbf{X}$ and $\sup_{x \in B_n} \overline{q}(x) \leq n$.

Condition II (W. Feller). For each $x \in \mathbf{X}$ the value of $\overline{q}(x)$ is finite: $\overline{q}(x) < \infty$.

Condition III (**local boundedness**). For each $s \in (T_0, T_1)$ and all $x \in \mathbf{X}$ the inequality $\sup_{t \in (T_0, s)} q(t, x) < \infty$ is true.

Condition IV (**local L^1-boundedness**). For each $s \in (T_0, T_1)$ and all $x \in \mathbf{X}$ we have $\int_{T_0}^{s} q(t, x)\, dt < \infty$.

Conditions I and II are equivalent, and, moreover,

$$\mathbf{I} \Longleftrightarrow \mathbf{II} \Longrightarrow \mathbf{III} \Longrightarrow \mathbf{IV}. \qquad (43)$$

In the case when the phase space $\mathbf{X}$ — is a discrete set $\mathbf{N}$, we proceeded from the fact that a Markov process has already been defined on some probability space; its transition probabilities are $p_{ij}(t)$ in the homogeneous case and $p_{ij}(s,t)$ in the inhomogeneous case.

From these transition probabilities, the Q-functions q_{ij} (in the homogeneous case) and $q_{ij}(t)$ (in the inhomogeneous case) were determined. In applied problems, as a rule, these **local** characteristics or $q_{ij}(t)$ are first specified, and then the transition probabilities are found from direct or inverse equations.

Note that in this case the Markov process itself, in its usual understanding, is not defined, and its Markov property is realized in the Kolmogorov-Chapman equation for transition functions.

Now, when considering the general case of jump Markov processes, it is appropriate to talk about how the corresponding Markov process can be constructed from Q-functions, and then, of course, about how the corresponding direct and inverse equations look for this process.

Following J. Jacod, we first introduce the space Ω, consisting of points

$$\omega = ((t_0, x_0), (t_1, x_1), \ldots), \tag{44}$$

defined as follows: $t_0 = T_0$, $x_0 \in \mathbf{X}$; if $t_n < T_1$, then $t_{n+1} > t_n$ и $x_n \in \mathbf{X}$, and if $t_n = T_1$,then $t_{n+1} = T_1$ и $x_n = x_\infty$, where x_∞ is an auxiliary state not belonging to $\mathbf{X}$. We will also denote $t_\infty(\omega) = \lim_n t_n(\omega)$. It is obvious, that Ω a measurable subset in $([T_0, T_1] \times \mathbf{X}')^\infty$, where $\mathbf{X}' = \mathbf{X} \cup \{x_\infty\}$. For such defined points $\omega \in \Omega$ we can construct an integer random measure

$$\mu(\omega; dt, dx) = \sum_{n \geq 1} I(t_n < T_1)\, \mathcal{E}_{(t_n, x_n)}(dt, dx), \tag{45}$$

where $t_n = t_n(\omega)$, $x_n = x_n(\omega)$ и $\mathcal{E}_a$ is the Dirac measure in the point a.

If on the space $(\Omega, \mathcal{F})$ where $\mathcal{F}$ is the sigma-algebra of measurable subsets a probability measure $\mathbf{P}$ is already given, then it is possible to construct a random measure

$$\nu(\omega; dt, dx) = \sum_{n \geq 1} \frac{I(t \leq t_{n+1})\, G_n(\omega; dt, dx)}{G_n(\omega; [t, T_1] \times \mathbf{X})}, \tag{46}$$

where

$$G_n(\omega; dt, dx) = \mathbf{P}(t_{n+1}(\omega) \in dt, x_{n+1}(\omega) \in dx \,|\, (t_0, x_0), \ldots, (t_n, x_n)) \tag{47}$$

and

$$G_n(\omega; [t, T_1] \times \mathbf{X}) = \mathbf{P}(t_{n+1}(\omega) \geq t \,|\, (t_0, x_0), \ldots, (t_n, x_n)) \tag{48}$$

(all probabilities on the right-hand sides are regular versions of the corresponding conditional probabilities).

This measure turns out to be predictable and such that for each set $B \in \mathcal{B}(\mathbf{X})$ the process

$$(\mu(\omega; (0, t \wedge t_n] \times B) - \nu(\omega; (0, t \wedge t_n] \times B))_{t \geq 0} \tag{49}$$

for each n is (with respect to the measure $\mathbf{P}$) a uniformly integrable martingale.

f the Q-function is given, then the measure ν can be constructed using the formula

$$\nu(\omega; [T_0, t] \times B) = \int_{T_0}^{t} \sum_{n \geq 0} I(t_n < s \leq t_{n+1}) q(s, x, B \setminus \{x_n\}) \, ds. \quad (50)$$

This measure is predictable, and then, according to the paper of J. Jacod (1975), on the space Ω described above with the corresponding sigma-algebra $\mathcal{F}$ it is possible to define such a measure $\mathbf{P}$, that property (49) is satisfied, and the process $X = (X_t)$, $t \in [T_0, T_1)$, where

$$X_t(\omega) = \sum_{n \geq 0} I(t_n(\omega) \leq t < t_{n+1}(\omega)) x_n(\omega) + I(t_\infty(\omega) \leq t) x_\infty(\omega) \quad (51)$$

will be a (purely) piecewise constant Markov process with a predetermined Q-function.

4.6 Following our works, we describe the results on inverse equations for the transition function $P(s, x; t, B)$ under a given Q-function. (Direct equations will be discussed later.)

By analogy with paragraph 3, let us put

$$\overline{P}^{(0)}(u, x; t, B) = I(x \in B) \exp\Big\{-\int_u^t q(x, s)\, ds\Big\} \tag{52}$$

and

$$\overline{P}^{(n)}(u, x; t, B) = \int_u^t \Big[\int_{\mathbf{X}} \exp\Big\{-\int_u^w q(x, \theta)\, d\theta\Big\}\, q(x, w; dy \setminus \{x\})\Big] \times \overline{P}^{(n-1)}(w, y; t, B)\, dw. \tag{53}$$

(Compare with formulas (33) in the case when the phase space is $\mathbf{N}$.)

Let

$$\overline{P}(u, x; t, B) = \sum_{n=0}^{\infty} \overline{P}^{(n)}(u, x; t, B). \tag{54}$$

In the works of Feller and ours it was shown that the function $\overline{P}$ (under condition IV) is a transition function such that $0 \leq \overline{P}(u, x; t, B) \leq 1$, and at the same time satisfies the Kolmogorov–Chapman equation.

It is also important that the process $X = (X_t)$, $t \in [T_0, T_1)$, will be a jump Markov process whose transition function is the function $\overline{P}(u, x; t, B)$.

To solve the question of what the inverse equation is for this transition function, we proceed as follows.

Let us introduce a family $\mathcal{P}$ of nonnegative transition functions $P(u,x;t,B)$ defined for all $t \in [T_0,T_1)$, $u \in [T_0,t)$, $x \in \mathbf{X}$, $B \in \mathcal{B}(\mathbf{X})$ and measurable by $(u,x) \in [T_0,t) \times \mathbf{X}$ for all $t \in [T_0,T_1)$. It is clear that the transition function $\overline{P}$ constructed above belongs to $\mathcal{P}$.

Теорема 1. *Let condition IV be satisfied. Then the function* $\overline{P}$ *is* ***minimal*** *among those functions* P *from* $\mathcal{P}$ *which for all* $t \in [T_0, T_1)$, $x \in \mathbf{X}$, $B \in \mathcal{B}(\mathbf{X})$ *have the following properties:*

(i) $\lim_{u \to t-} P(u, x; t, B) = I(x \in B)$;

(ii) P *is absolutely continuous with respect to* $u \in [T_0, t)$;

(iii) P *satisfies the inverse equation*

$$\frac{\partial}{\partial u} P(u, x; t, B) = q(x, u) P(u, x; t, B) - \int_{\mathbf{X}} q(x, u, dy \setminus \{x\}) P(u, y; t, B) \tag{55}$$

for almost all $u \in [T_0, t)$.

If the transition function $\overline{P}$ *is probabilistic (i.e.* $\overline{P}(u, x; t, B) = 1$ *for all* u, x , t *from the domain of definition of* $\overline{P}$*), then it is* ***unique*** *among all those* $P \in \mathcal{P}$ *that satisfy the properties* ***(i)****–****(iii)*** *and take values in* $[0, 1]$.

The case of direct equations is more complicated. This is largely due to the fact that the paths of the observed process can “explode” both at a deterministic and at a random moment in time. In the case of inverse equations, we start from a predetermined moment T_0.

Let us give some definitions.

Let us define $\widehat{\mathcal{P}}$, i.e. a family of real-valued functions $\widehat{P}(u, x; t, B)$ defined for all $u \in [T_0, T_1)$, $t\ in[u, T_1)$, $x \in \mathbf{X}$, $B \in \mathcal{B}(\mathbf{X})$ and which are measures on $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$ for fixed u, x, t and measurable by t functions for fixed u, x, B. (Compare with the definition of class $\mathcal{P}$.)

Note that the transition function $\overline{P}$ introduced by the formula (54) belongs to $\widehat{\mathcal{P}}$.

We will also need the following concept.

Definition 2. Let $s \in [T_0, T_1)$. We will say that the set $B \in \mathcal{B}(\mathbf{X})$ is (q, s)-bounded if the function $q(x, t)$ is bounded on the set $B \times [T_0, s)$.

In the case $s = T_1$ we call the set $B \in \mathcal{B}(\mathbf{X})$ q-bounded.

Remarque 2. The definition of (q, s)-bounded sets is introduced because the right-hand side of the direct equation (see further (56)) may turn out to have the form $\infty - \infty$. This complication is absent for inverse equations, since under condition IV the right-hand side of the inverse equation (55) for almost all $u \in (T_0, s)$ is either a real number or equal to $-\infty$.

If $q(u, x) < \infty$ for all $x \in \mathbf{X}$ and $u \in (T_0, s)$, then any one-point set $B = \{x\}$ is (q, s)-bounded. Since in the case of processes with a countable set of states, Kolmogorov's equations are written only for single-point sets, then in this case the complexity mentioned above is absent.

Thus, the definition of (q, s)-bounded sets is essential only for direct equations in the case of an uncountable set of states.

Теорема 2. *Let condition III be satisfied. Then the function* $\overline{P}$ *is minimal among those* P *from* $\widehat{\mathcal{P}}$ *which for all* $u \in [T_0, T_1)$, $s \in (u, T_1)$, $x \in \mathbf{X}$ *and all* (q, s)*-bounded sets* $B \in \mathcal{B}(\mathbf{X})$ *have the following properties:*

(i) $\lim_{t \to u+} P(u, x; t, B) = I(x \in B)$;

(ii) P *s absolutely continuous with respect to* $t \in [u, s)$;

(iii) P *satisfies the direct equation*

$$\frac{\partial}{\partial t} P(u, x; t, B) = -\int_B q(y, t) P(u, x; t, dy) + \int_{\mathbf{X}} q(y, t, B \setminus \{y\}) \, P(u, x; t, dy \tag{56}$$

for almost all $t \in (u, s)$.

If additionally the transition function $\overline{P}$ *is probabilistic, then it is* ***unique*** *among those* $P \in \widehat{\mathcal{P}}$ *that satisfy the properties* ***(i)–(iii)*** *and take values in* $[0, 1]$.