A Tale of Three Coauthors: Comparison of Ising Models

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$$\langle f\rangle_{\mu,\Lambda} = Z^{-1}\langle fe^{-H}\rangle_{\mu,0}; \qquad Z = \langle e^{-H}\rangle_{\mu,0}$$

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$$\langle \sigma^B \rangle_{\mu,\Lambda} \leq \langle \sigma^B \rangle_{\nu,\Lambda}$$

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$$T_c(\mu) \geq T_-(\mu)^2 T_c(\textit{classical Ising})$$

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The left hand side is an Ising expectation and the right with the apriori measure of the $2D$ rotor with only couplings of the 1 components. So this was part of what seems to be an Ising domination result (the 2 indicates the Ising measure should really be $b_{1/\sqrt{2}}$).

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We will see shortly that $(G2) \Rightarrow (G1)$

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Ginibre Theorem 1 *If a triple* $\langle X, \mu, \mathcal{F} \rangle$ *obeys* $(G2)$, *so does* $\langle X, \mu, \mathcal{C}(\mathcal{F}) \rangle$.

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What makes the notion so powerful is that there are three theorems for getting new Ginibre systems from old ones.

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The proof is easy. The normalization is irrelevant and we expand the exponential $\exp(-H(x) - H(y))$.

Classical Ising System

Ginibre Theorem 4 *Let X be $\mathbb{R}$ or a compact subset of the form $[-A, A]$ and let $d\mu$ be a probability measure which is invariant under $x \mapsto -x$ and so that (only non-trivial in case X is not compact) $\int x^{2n}\, d\mu(x) < \infty$ for all n. Let $\mathcal{F}$ contain the single function, $f(x) = x$. Then $\langle X, \mu, \mathcal{F} \rangle$ is a Ginibre system.*

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Interchanging x and y implies the integral is zero if m is odd and $x \mapsto -x$ symmetry implies the integral is zero if $m+k$ is odd. Thus the only possible non-zero integrals are when m and k are even in which case the integrand is positive!

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$$-H = \sum_{A \subset \Lambda} J(A) \sigma^A \qquad \sigma^A = \prod_{j \in A} \sigma_j$$

with ANY (!!!) even apriori measure, one has positive expectations and positive correlations of the σ^A.

Final Ginibre Thoughts

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Basic Definition

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$$\langle \sigma^A \rangle_{\mu_H} \leq \langle \sigma^A \rangle_{\nu_H}$$

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Big Theorem *Let $d\mu$ be an even probability measure on $\mathbb{R}$ with compact support that is not a point mass at 0.*

Statement of the Theorem

We say an even probability measure is non-trivial if and only if it is not a unit mass at 0. The following theorem says that any non-trivial measure of compact support is Ising dominated by a scaling of any other such measure and gives quantitative optimal bounds when one of the measures is the Bernoulli measure.

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$$S \leq T_- \iff \forall_{n\in\mathbb{N}} \int_{\mathbb{R}} (x^2 - S^2)^n \, d\mu(x) \geq 0$$

What is T_-

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It is an interesting question when one has equality. Before leaving this theorem, I should mention I happened to look at a 1981 paper of Bricmont, Lebowitz and Pfister that includes in an appendix a proof (with attribution to Wells) of Wells result about the existence of $T_- > 0$.

Three Spin Values

For $0 \leq \lambda \leq 1$, consider the probability measure supported by the three points $\{0, \pm 1\}$ given by

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$$T_{-}(\lambda) = \begin{cases} \sqrt{\lambda}, & \text{if } \lambda \leq \frac{1}{2} \\ \sqrt{\frac{1}{2}}, & \text{if } \lambda \geq \frac{1}{2} \end{cases}$$

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So we see there are cases where $T_{-} = \langle x^2 \rangle^{1/2} = \sqrt{\lambda}$ and other cases where the inequality is strict. Note also that at $\lambda = \frac{1}{2}$, the integral $\langle (x^2 - T_{-}^2)^{2m+1} \rangle_\lambda$ vanishes for all n, a sign that the distribution of $x^2 - T_{-}^2$ is symmetric about 0.

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Conjecture For $S = \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots$ one has that

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Totally Anisotropic D-vector model

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As explained earlier, because Wells domination implies Ising domination, one has that for pair interactions

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Not only is this an improvement of Griffiths by more than $\frac{4}{3}$ but in the result for $S \neq 1$, the improved constant is optimal!! For one has equality if T_c is replaced by its mean field values and as noted by Dyson, Lieb and Simon, mean field theory is exact in the nearest neighbor infinite dimension limit.

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By rescaling so the maximum spin value is S, the conjecture is the assertion that for $m = 1, 2, \ldots$ and $S = \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots$

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For S an integer, this is the usual kind of sum.

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$$\sum_{j=-S}^{S} (3j^2 - S(S+1))^{2m+1} \geq 0$$

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It seemed a shame not to make one more push to prove the conjecture so I did the obvious thing.

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His note had one wonderful idea (using Karamata's inequality) and 5 dense pages of calculation to implement it.

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José also suggested it would be good to try again to locate Daniel Wells.

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Majorization

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If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_{+,\geq}$ (the set with $x_1 \geq x_2 \geq \ldots x_n \geq 0$), we say that $\mathbf{x}$ *majorizes* $\mathbf{y}$, written $\mathbf{x} \succ \mathbf{y}$ or $\mathbf{y} \prec \mathbf{x}$ if an only if

$$\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j; \quad S_k(\mathbf{x}) \equiv \sum_{j=1}^{k} x_j \geq \sum_{j=1}^{k} y_j,\ k = 1, \ldots, n-1$$

which defines $S_k(\mathbf{x})$.

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Majorization

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The key fact is that $\mathbf{y} \prec \mathbf{x}$ iff y is in the convex hull in $\mathbb{R}^n$ of the (at most) n! points obtained from x by permuting the coordinates proven by slicing $\mathbb{R}^n$ with specific hyperplanes.

Karamata's Inequality

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Theorem (Karamata's Inequality) Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_{+,\geq}$ with $\mathbf{x} \succ \mathbf{y}$ and let φ be an arbitrary continuous convex function on $[0, x_1]$. Then

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Even though this is widely referred to as Karamata's inequality after Karamata's 1932 paper, it or theorems that imply it appear in a 1923 paper of Schur and a 1929 paper of Hardy-Littlewood-Pólya.

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That said, we note that HLP doesn't have a proof which may not have appeared until their 1934 book and that Karamata proved a converse, namely, if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_{+,\geq}$ and the inequality holds for all convex φ, then $\mathbf{x} \succ \mathbf{y}$.

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The proof of Karamata's theorem is simple. One proves the convex hull result and then one notes the function $\mathbf{w} \mapsto \sum_{j=1}^{n} \varphi(w_j)$ is convex and permutation symmetric.

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which is true by convexity of ψ.

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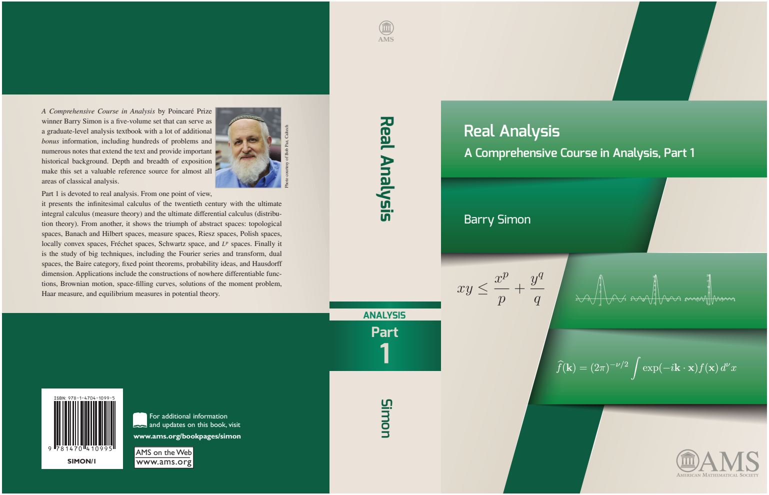

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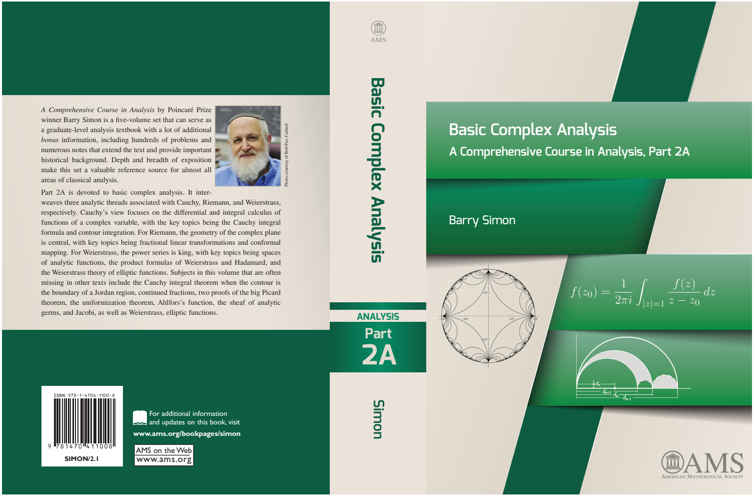

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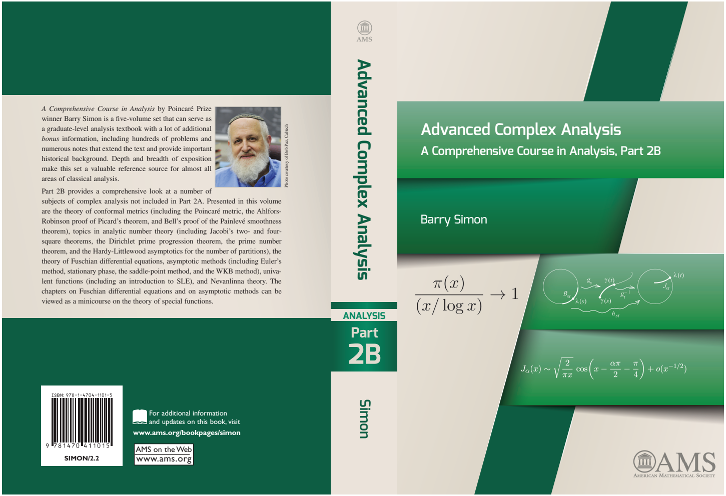

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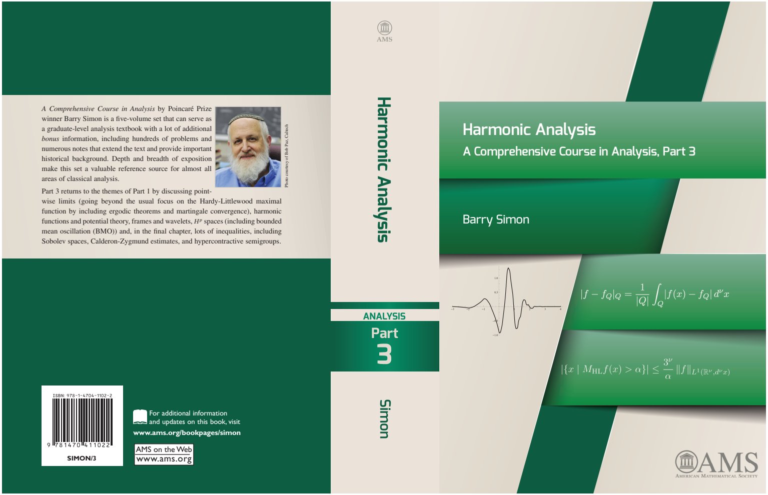

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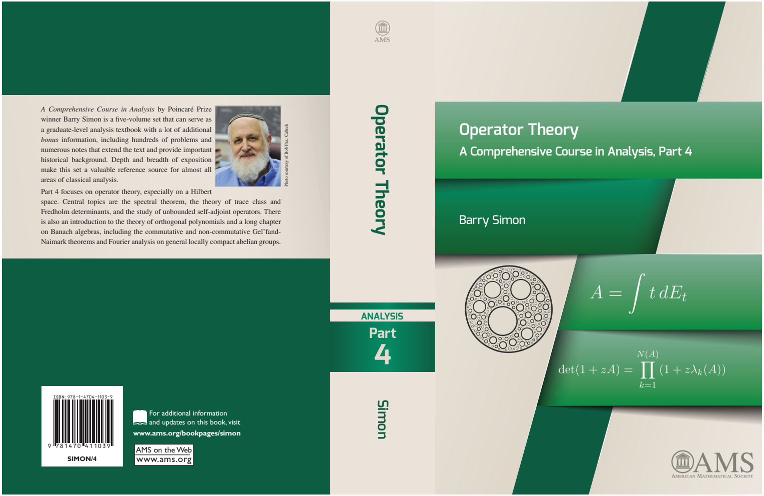

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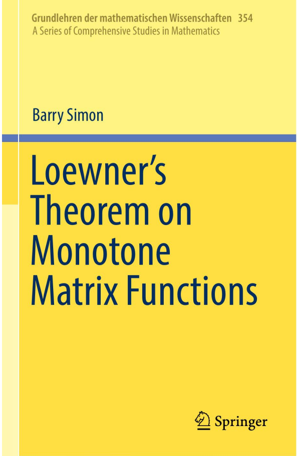

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