Wavelet approximation in Orlicz spaces

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joint results with **A. Krivoshein**

A multivariatre wavelet system generated by functions $\psi^{(l)}$, $l = 1, \dots, r$ (called wavelet functions), is

$$\{\psi_{ik}^{(l)}\}_{i,k,l},$$

where

$$\psi_{jk}^{(l)}(x) := m^{j/2}\psi^{(l)}(M^j x + k), \quad j \in \mathbb{Z}, k \in \mathbb{Z}^d$$

M is a $d \times d$ integer matrix whose eigenvalues are bigger than 1 in absolute value (called matrix dilation) and $m = |\det M|$.

We say that a wavelet system $\{\psi_{ik}^{(l)}\}_{i,k,l}$ has VM^s property (vanishing moment property of order s) if

$$D^\beta\widehat{\psi^{(l)}}(\mathbf{0}) = 0 \ \ \forall\beta \in \mathbb{Z}^d : 0 \le \|\beta\|_1 < s, \quad l = 1, \dots, r.$$

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We discuss approximation properties in Orlicz spaces of decompositions with respest to

1. wavelet frames/Riesz bases
2. frame-like wavelet systems

Let $\mathcal{H}$ be a Hilbert space. A system $\{f_n\}_{n=1}^{\infty} \subset \mathcal{H}$ is called a frame if there exist $A, B > 0$ such that

$$A\|f\|^2 \le \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \le B\|f\|^2$$

for all $f \in \mathcal{H}$. If only the right-hand inequality is satisfied for all $f \in \mathcal{H}$, then $\{f_n\}_{n=1}^{\infty}$ is called a Bessel system. An important property of a frame is the following: every $f \in \mathcal{H}$ can be decomposed as

$$f = \sum_{n=1}^{\infty} \langle f, \widetilde{f_n}\rangle f_n,$$

where $\{\widetilde{f_n}\}_{n=1}^{\infty}$ is a dual frame in $\mathcal{H}$.

If $A = B$ then the frame is tight.

A tight frame $\{f_n\}_{n=1}^{\infty}$ coincides with its dual frame $\{\widetilde{f_n}\}_{n=1}^{\infty}$.

Dual wavelet frames

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A tight frame $\{f_n\}_{n=1}^{\infty}$ coincides with its dual frame $\{\widetilde{f_n}\}_{n=1}^{\infty}$.

If $\{\psi_{ik}^{(l)}\}_{i,k,l}$, $\{\widetilde{\psi}_{ik}^{(l)}\}_{i,k,l}$ are dual wavelet frames in $L_2(\mathbb{R}^d)$ generated by wavelet functions $\psi^{(l)}, \widetilde{\psi}^{(l)}$, $l = 1, \dots, r$, then every $f \in L_2(\mathbb{R}^d)$ can be decomposed as

$$f = \sum_{i=-\infty}^{\infty} \sum_{k \in \mathbb{Z}^d} \sum_{l=1}^{r} \langle f, \widetilde{\psi}_{ik}^{(l)} \rangle \psi_{ik}^{(l)}.$$

Usually wavelet frames are constructed in framework of multiresolution analysis from dual scaling functions φ, $\widetilde{\varphi}$. In this case the wavelet decomposition may be written also in the form

$$\langle f, \widetilde{\varphi}(\cdot + k) \rangle \varphi(\cdot + k) + \sum_{i=0}^{\infty} \sum_{k \in \mathbb{Z}^d} \sum_{l=1}^{r} \langle f, \widetilde{\psi}_{ik}^{(l)} \rangle \psi_{ik}^{(l)}$$

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For an apropriate pair of scaling functions φ, $\widetilde{\varphi}$ (satiafying some very special properties), there exists a method, called MEP (matrix extensial principle), providing a dual wavelet system $\{\psi_{ik}^{(l)}\}_{i,k,l}$, $\{\widetilde{\psi}_{ik}^{(l)}\}_{i,k,l}$ and we have

$$\sum_{k\in\mathbb{Z}^d}\langle f,\widetilde{\varphi}_{0k}\rangle\varphi_{0k}+\sum_{i=0}^{j-1}\sum_{k\in\mathbb{Z}^d}\sum_{l=1}^{r}\langle f,\widetilde{\psi}_{ik}^{(l)}\rangle\psi_{ik}^{(l)}=\sum_{k\in\mathbb{Z}^d}\langle f,\widetilde{\varphi}_{jk}\rangle\varphi_{jk}$$

If both the systems $\{\psi_{ik}^{(l)}\}_{i,k,l}$ and $\{\widetilde{\psi}_{ik}^{(l)}\}_{i,k,l}$ are Bessel, than they form dual wavelet frames.

The simplest example is the Haar system $\{\psi_{ik}\}_{i,k}=\{\widetilde{\psi}_{ik}\}_{i,k}$, that is generated from the scaling function $\varphi=\widetilde{\varphi}=\chi_{[0,1]}$ by MEP. This system is an orthonormal basis.

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A function $\theta : [0, \infty] \to [0, +\infty]$ is called a Young function, if it is convex and $\theta(x) = 0$ if and only if $x = 0$.
For a Young function θ, denote

$$I_\theta(f) := \int_{\mathbb{R}^d} \theta(|f(x)|)dx.$$

$$L_\theta := L_\theta(\mathbb{R}^d) := \{f \text{ measurable on } \mathbb{R}^d : I_\theta(\lambda f) < \infty \text{ for some } \lambda > 0\};$$

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A sequence $\{f_n\}_n \subset L_\theta$ $(\subset E_\theta)$ is said to modular converge to zero in L_θ (in E_θ) if $I_\theta(\lambda f_n) \to 0$ for some λ (for all λ).

A Young function θ is said to satisfy Δ_2-condition if there exists a constant $K > 2$ such that

$$\theta(2x) \le K\theta(x), \quad \forall x \ge 0.$$

This condition is necessary and sufficient for $L_\theta = E_\theta$.

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The Young functions $\theta(x) = x^p$, $\theta(x) = x^p(|\ln x| + 1)$, $p \geq 1$, $\theta(x) = x^p \ln^\alpha(e + x)$, $p \geq 1$, $\alpha \geq 0$, and $\theta(x) = (1 + x)\ln(1 + x) - x$ satisfy Δ_2-condition.

The Young functions $\theta(x) = e^x - 1$, $\theta(x) = e^{x^2} - 1$ and $\theta(x) = e^x - x - 1$ do not satisfy Δ_2-condition

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$\sum_{k\in\mathbb{Z}^d}\langle f,\widetilde{\varphi}_{jk}\rangle\varphi_{jk}$, $d=1$, $\widetilde{\varphi}=\chi_{[0,1]}$

$\forall f\in L_\theta \quad \exists\lambda>0:$

$$I_\theta\left(\lambda\left(f-\sum_{k\in\mathbb{Z}^d}\langle f,\widetilde{\varphi}_{jk}\rangle\varphi_{jk}\right)\right)\longrightarrow 0,\quad j\to\infty$$

We are interested in the approximation order of wavelet expansion, i.e. the decay rate of the error $I_\theta\left(\lambda\left(f-\sum_{k\in\mathbb{Z}^d}\langle f,\widetilde{\varphi}_{jk}\rangle\varphi_{jk}\right)\right)$

W^s_θ denotes the Orlicz-Sobolev space of order $s\in\mathbb{N}$, i.e.

$$W^s_\theta=\{f\ \textit{measurable}:\ D^\beta f\in L_\theta,\quad 0\le\|\beta\|_1\le s\}.$$

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Theorem

Let $s \in \mathbb{N}$, $f \in W_\theta^s$. *Suppose* $\{\psi_{jk}^{(l)}\}_{j,k,l}$, $\{\widetilde{\psi}_{jk}^{(l)}\}_{j,k,l}$ *are dual wavelet frames generated from* φ, $\widetilde{\varphi}$ *by MEP,* $\{\varphi_{0k}\}_k$, $\{\widetilde{\varphi}_{0k}\}_k$ *are Bessel systems,* $\widehat{\varphi}(\mathbf{0}) = \widehat{\widetilde{\varphi}}(\mathbf{0}) = 1$ *and the system* $\{\widetilde{\psi}_{jk}^{(l)}\}_{j,k,l}$ *has* VM^s *property. If*

$$|\varphi(x)|, |\widetilde{\varphi}(x)|, |\psi^{(l)}(x)|, |\widetilde{\psi}^{(l)}(x)| \le \nu(x) \quad \forall x \in \mathbb{R}^d,$$

where ν *is a radial decreasing function such that* $|\cdot|^s \nu \in L_1 \cap L_\infty$. *Then for any number* $\rho > 1$ *which is less than any eigenvalue of* M *in absolute value, there exists* $\lambda > 0$ *such that*

$$I_\theta\left(\lambda\left(f - \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk}\right)\right) \le C\rho^{-sj} \sum_{\|\beta\|_1 = s} I_\theta(\lambda_\beta\ D^\beta f), \quad (*)$$

where C *does not depend on* f *and* j, *i.e., the wavelet expansion has approximation order* s *in the sense of modular convergence.*
If, moreover, $f \in E_\theta$ *and* $D^\beta f \in E_\theta$, $\|\beta\|_1 = s$, *then (*) holds for every* $\lambda > 0$.

Example

Let $\theta(x) = e^x - 1$ (does not satisfy Δ_2-condition), $d = 1$, $M = 2$, $s = 1$, $\varphi = \widetilde{\varphi} = \chi_{[0,1]}$, $f(x) = \max\{0, \frac{1}{2} - |x|\}$.

$I_\theta(f) < \infty$, $I_\theta(f') < \infty \quad \Rightarrow f \in W^1_\theta$.

The wavelet system $\{\psi_{jk}\}$ generated from φ by MEP is the Haar basis, and for every $\lambda > 0$

$$\sum_{k\in\mathbb{Z}} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk}(x) = \sum_{k=-2^{j-1}}^{2^{j-1}-1} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk}(x) = \frac{1}{2} - \frac{|2k+1|}{2^{j+1}} \text{ for } x \in \left[\frac{k}{2^j}, \frac{k+1}{2^j}\right], \ j > 1;$$

$$I_\theta\left(\lambda\big(f - \sum_{k\in\mathbb{Z}} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk}(x)\big)\right) = \int\limits_{[-\frac{1}{2},\frac{1}{2}]} \left(e^{\lambda|f(x) - \sum\limits_{k\in\mathbb{Z}} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk}(x)|} - 1\right) dx$$

$$= \sum_{k=-2^{j-1}}^{2^{j-1}-1} \int\limits_{[\frac{k}{2^j}, \frac{k+1}{2^j}]} \left(e^{\lambda\left\|\frac{1}{2} - |x| - \left(\frac{1}{2} - \frac{|2k+1|}{2^{j+1}}\right)\right|} - 1\right) dx = \sum_{k=-2^{j-1}}^{2^{j-1}-1} \int\limits_{[0, \frac{1}{2^j}]} \left(e^{\lambda\left||\frac{k}{2^j} + \frac{1}{2^{j+1}}| - |\frac{k}{2^j} + x|\right|} - 1\right) dx$$

$$\geq \sum_{k=0}^{2^{j-1}-1} \int\limits_{[0, \frac{1}{2^{j+1}}]} \left(e^{\frac{\lambda}{2^{j+1}} - \lambda x} - 1\right) dx \geq \frac{1}{16} \frac{\lambda}{2^j}.$$

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$$I_\theta\left(\lambda(f - \sum_{k\in\mathbb{Z}} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk}(x))\right) = \int_{[-\frac{1}{2},\frac{1}{2}]} \left(e^{\lambda|f(x) - \sum\limits_{k\in\mathbb{Z}} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk}(x)|} - 1\right) dx$$

$$= \sum_{k=-2^{j-1}}^{2^{j-1}-1} \int_{[\frac{k}{2^j}, \frac{k+1}{2^j}]} \left(e^{\lambda\left\| \frac{1}{2} - |x| - \left(\frac{1}{2} - \frac{|2k+1|}{2^{j+1}}\right)\right|} - 1\right) dx = \sum_{k=-2^{j-1}}^{2^{j-1}-1} \int_{[0,\frac{1}{2^j}]} \left(e^{\lambda\left| |\frac{k}{2^j} + \frac{1}{2^{j+1}}| - |\frac{k}{2^j} + x|\right|} - 1\right) dx$$

$$\geq \sum_{k=0}^{2^{j-1}-1} \int_{[0,\frac{1}{2^{j+1}}]} \left(e^{\frac{\lambda}{2^{j+1}} - \lambda x} - 1\right) dx \geq \frac{1}{16}\frac{\lambda}{2^j}.$$

The error estimate (*) for the modular convergence may be improved as follows

$$I_\theta\left(\lambda\left(f-\sum_{k\in\mathbb{Z}^d}\langle f,\widetilde{\varphi}_{jk}\rangle\varphi_{jk}\right)\right)\le C\sum_{\|\beta\|_1=s} I_\theta\left(\frac{\sum\limits_{i=j}^{\infty}\|M^{-i}\|^s}{\sum\limits_{i=0}^{\infty}\|M^{-i}\|^s}\lambda_s\ D^\beta f\right),$$

Let $\theta(x)=e^{x^2}-1$ (does not satisfy Δ_2-condition), $d=1,\ M=2,\ s=1$.

$$I_\theta\left(\frac{\sum\limits_{i=j}^{\infty}2^{-i}}{\sum\limits_{i=0}^{\infty}2^{-i}}\lambda_1 f'\right)=\int_{-\infty}^{\infty}\left(e^{2^{-2j}|\lambda_1 f|^2}-1\right)\le 2^{-2j}I_\theta(\lambda_1 f')$$

Thus the approximation order is 2 in this case.

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Thus the approximation order is 2 in this case.

$$\|f\|_\theta := \inf\left\{\gamma > 0 : \; I_\theta\left(\frac{f}{\gamma}\right) \le 1\right\}, \quad f \in L_\theta.$$

This functional is well defined on L_θ, $\|\cdot\|_\theta$, and it is a norm in L_θ

(called the Luxemburg norm).

The normed space $(L_\theta, \|\cdot\|_\theta)$ is a Banach space.

Theorem

Let θ *satisfy* Δ_2*-condition,* $s \in \mathbb{N}$*,* $f \in W^s_\theta$*. Suppose* $\{\psi^{(l)}_{jk}\}_{j,k,l}$*,* $\{\widetilde{\psi}^{(l)}_{jk}\}_{j,k,l}$ *are dual wavelet frames generated from* φ*,* $\widetilde{\varphi}$ *by MEP,* $\{\varphi_{0k}\}_k$*,* $\{\widetilde{\varphi}_{0k}\}_k$ *are Bessel systems,* $\widehat{\varphi}(\mathbf{0}) = \widehat{\widetilde{\varphi}}(\mathbf{0}) = 1$ *and the system* $\{\widetilde{\psi}^{(l)}_{jk}\}_{j,k,l}$ *has* VM^s *property.*

If

$$|\varphi(x)|, |\widetilde{\varphi}(x)|, |\psi^{(l)}(x)|, |\widetilde{\psi}^{(l)}(x)| \le \nu(x) \quad \forall x \in \mathbb{R}^d,$$

where ν *is a radial decreasing function such that* $|\cdot|^s \nu \in L_1 \cap L_\infty$*, then*

$$\left\| f - \sum_{k \in \mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk} \rangle \varphi_{jk} \right\|_\theta \le C \rho^{-sj} \|f\|_{W^s_\theta},$$

where $\rho > 1$ *is any number that is less than any eigenvalue of* M *in absolute value and* C *does not depend on* f *and* j*, i.e., the wavelet expansion has approximation order* s *in the sense of convergence in the Luxemburg norm.*

Dual frame-like wavelet systems

Using the construction of dual wavelet frames by MEP, one has to overcome substantial difficulty to provide vanishing moments for all wavelet functions $\psi^{(l)}, \widetilde{\psi}^{(l)}$, which is a necessary condition for the systems $\{\psi^{(l)}_{ik}\}_{i,k,l}$, $\{\widetilde{\psi}^{(l)}_{ik}\}_{i,k,l}$ to be frames in $L_2(\mathbb{R}^d)$.

However, engineers often do not take care of this. Providing vanishing moments only for the functions $\widetilde{\psi}^{(l)}$, they successfully apply such "frames" (which are really not frames) for signal processing. Thus, it makes sense to study a wider class of dual wavelet systems which preserve the frame-type decompositions.

A method for the construction compactly supported dual wavelet systems $\{\psi^{(\nu)}_{jk}\}_{j,k,l}$, $\{\widetilde{\psi}^{(l)}_{jk}\}_{j,k,l}$, where $\psi^{(l)}, \widetilde{\psi}^{(l)}$ are, generally speaking, tempered distributions and the system $\{\widetilde{\psi}^{(l)}_{jk}\}_{j,k,l}$ has VM^s property, was developed in

A. Krivoshein and M. Skopina, Approximation by frame-like wavelet systems, Appl. Comput. Harmon. Anal. 31 (2011), no 3, 410–428

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A. Krivoshein and M. Skopina, Approximation by frame-like wavelet systems, Appl. Comput. Harmon. Anal. 31 (2011), no 3, 410-428

Using the construction of dual wavelet frames by MEP, one has to overcome substantial difficulty to provide vanishing moments for all wavelet functions $\psi^{(l)}, \widetilde{\psi}^{(l)}$, which is a necessary condition for the systems $\{\psi_{ik}^{(l)}\}_{i,k,l}$, $\{\widetilde{\psi}_{ik}^{(l)}\}_{i,k,l}$ to be frames in $L_2(\mathbb{R}^d)$.

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A compactly supported tempered distribution φ is called refinable, if it satisfies a refinement equation

$$\widehat{\varphi}(\xi) = m_0(M^{*-1}\xi)\widehat{\varphi}(M^{*-1}\xi),$$

where m_0 (called refinable mask=scaling mask) is a trigonometric polynomial.

The polyphase components of m_0 are the trigonometric polynomials μ_{0k} defined by

$$m_0(x) = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} e^{2\pi i (s_k, x)} \mu_{0k}(M^* x).$$

where $s_0, \dots, s_{m-1}$ are digits of M (for instance, the set of digits can be taken as $M[0,1)^d \cap \mathbb{Z}^d$).

Starting with two arbitrary trigonometric polynomial m_o and $\widetilde{m}_0$, we can construct two refinable functions

$$\widehat{\varphi}(\xi) := \prod_{j=1}^{\infty} m_0(M^{*-j}\xi), \quad \widehat{\widetilde{\varphi}}(\xi) := \prod_{j=1}^{\infty} \widetilde{m}_0(M^{*-j}\xi)..$$

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Given scaling masks m_0, $\widetilde{m}_0$, satisfying $m_0(\mathbf{0}) = \widetilde{m}_0(\mathbf{0}) = 1$, find trigonometric polynomials (called wavelet masks) $m_l, \widetilde{m}_l$, $l = 1, \dots, r$, $r \geq m-1$, such that the corresponding polyphase matrices

$$\mathcal{M} := \begin{pmatrix} \mu_{00} & \cdots & \mu_{0,m-1} \\ \vdots & \ddots & \vdots \\ \mu_{r,0} & \cdots & \mu_{r,m-1} \end{pmatrix}, \widetilde{\mathcal{M}} := \begin{pmatrix} \widetilde{\mu}_{00} & \cdots & \widetilde{\mu}_{0,m-1} \\ \vdots & \ddots & \vdots \\ \widetilde{\mu}_{r,0} & \cdots & \widetilde{\mu}_{r,m-1} \end{pmatrix}$$

satisfy

$$\mathcal{M}^T \overline{\widetilde{\mathcal{M}}} = I_m,$$

and define wavelet functions by

$$\widehat{\psi^{(l)}}(\xi) = m_l(M^{*-1}\xi)\widehat{\varphi}(M^{*-1}\xi), \quad \widehat{\widetilde{\psi}^{(l)}}(\xi) = \widetilde{m}_l(M^{*-1}\xi)\widehat{\widetilde{\varphi}}(M^{*-1}\xi).$$

If

$$D^{\beta}\mu_{0k}(\mathbf{0}) = \frac{1}{\sqrt{m}} \sum_{\mathbf{0}\le\gamma\le\beta} \lambda_\gamma \binom{\beta}{\gamma} (-2\pi i M^{-1} s_k)^{\beta-\gamma}$$

for all $\beta \in \mathbb{Z}_+^d$ such that $\|\beta\|_1 < s$, $k = 0, \dots, m-1$ and for some complex numbers λ_γ, $\gamma \in \mathbb{Z}_+^d$, $\|\gamma\|_1 < s$, $\lambda_{\mathbf{0}} = 1$, and

$$D^{\beta}\left(1 - \sum_{k=0}^{m-1} \mu_{0k}(\xi)\overline{\widetilde{\mu}_{0k}(\xi)}\right)\Bigg|_{\xi=\mathbf{0}} = 0 \quad \forall \beta \in \mathbb{Z}_+^d, \|\beta\|_1 < s.$$

Then there exist associated wavelet functions $\psi^{(l)}, \widetilde{\psi}^{(l)}, l = 1, \dots, m$, such that the wavelet functions $\{\widetilde{\psi}^{(l)}\}_{j,k,l}$ have vanishing moments up to order s, i.e. $D^{\beta}\widehat{\widetilde{\psi}^{(l)}} = 0$ for all $\beta \in \mathbb{Z}_+^d$ such that $\|\beta\|_1 < s$.

What happens in L_p?

If $f, \varphi \in L_p$, $\widetilde{\varphi} \in L_q$, $1/p + 1/q = 1$, $p \geq 1$, then

$$\sum_{k \in \mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk} \rangle \varphi_{jk} \in L_p,$$

and if moreover $\{\widetilde{\psi}_{jk}^{(l)}\}_{j,k,l}$ has VM^s property and $f \in W_p^s$, then

$$\left\| f - \sum_{k \in \mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk} \rangle \varphi_{jk} \right\|_p \leq C \rho^{-sj} \|f\|_{W_p^s},$$

where $\rho > 1$ is any number that is less than any eigenvalue of M in absolute value and C does not depend on f and j.

The function θ^* given as

$$\theta^*(t) = \sup_{s \geq 0}(st - \theta(s)), \quad t \geq 0$$

is called conjugate to the Young function θ.

Let θ be a Young function and θ^* be its conjugate. If $f \in L_\theta$ and $g \in L_{\theta^*}$, then

$$\int_{\mathbb{R}^d} |fg| \, d\mu \leq 2\|f\|_\theta \|g\|_{\theta^*}.$$

If $f, \varphi \in L_\theta$, $\widetilde{\varphi} \in L_{\theta^*}$, then $\sum_{k \in \mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk} \rangle \varphi_{jk} \in L_\theta$???

A Young function θ is said to satisfy Δ'-condition if there exist $c_0, x_0 > 0$ such that $\theta(xy) \leq c_0 \theta(x)\theta(y)$ for all $y \geq 0$ and for all $x \geq x_0$.

Δ'-condition implies Δ_2-condition

What happens in the Orlicz spaces?

The function θ^* given as

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A Young function θ is said to satisfy Δ'-condition if there exist $c_0, x_0 > 0$ such that $\theta(xy) \leq c_0\theta(x)\theta(y)$ for all $y \geq 0$ and for all $x \geq x_0$.
Δ'-condition implies Δ_2-condition

Example

Let $\theta(s) = s\ln(e+s)$, (satisfies Δ'-condition) θ^* be conjugate to θ.

$$\widetilde{\varphi}(x) = \begin{cases} \frac{|\ln x|}{2} & x \in [0, 1/10] \\ 0 & \text{otherwise} \end{cases} \qquad \widetilde{\varphi} \in L_{\theta^*} \ ?$$

Let $t > 1$, $\quad g(s) := st - \theta(s) = s(t - \ln(e+s))$, $\;g(s^*) = \sup_{s \geq 0} g(s)$

$g'(0) > 0$ and $g'(s) < 0$ whenever $\ln(e+s) > t$

$\Rightarrow \ln(e+s^*) \leq t$

$\Rightarrow \theta^*(t) = s^*(t - \ln(e+s^*)) \leq (e^t - e)(t - \ln(e+s^*)) \leq te^t$

$$\int_{-\infty}^{\infty} \theta^*\left(\widetilde{\varphi}(x)\right) dx = \int_0^{1/10} \theta^* \left(\frac{|\ln(x)|}{2} \right) dx$$

$$\leq \int_0^{1/10} \frac{1}{2} |\ln x| e^{\frac{1}{2}|\ln x|} dx = -\frac{1}{2} \int_0^{1/10} \frac{\ln x}{\sqrt{x}} dx < \infty.$$

$$\varphi = \chi_{[0,1]}, \quad f(x) = \begin{cases} \frac{1}{\ln^2 n} & x \in [n, n+\frac{1}{n}], n \geq 10, \\ 0 & \text{otherwise}. \end{cases}$$

Obviously, $\varphi, f \in L_\theta$

$$I_\theta\Big(\sum_k \langle f, \widetilde{\varphi}_{0k}\rangle \varphi_{0k}\Big) = \int_{\mathbb{R}} \theta \left(\sum_k \int_{\mathbb{R}} f(t)\widetilde{\varphi}(t+k)dt \;\varphi(x+k)\right) dx$$

$$= \sum_n \int_0^1 \theta \left(\sum_k \int_{\mathbb{R}} f(t)\widetilde{\varphi}(t+k)dt \;\varphi(x+k+n)\right) dx$$

$$= \sum_n \theta \left(\int_{\mathbb{R}} f(t)\widetilde{\varphi}(t-n)dt \right) = \sum_{n=10}^{\infty} \theta \left(\int_0^1 f(t+n)\widetilde{\varphi}(t)dt \right).$$

$$= \sum_{n=10}^{\infty} \theta \left(\frac{1}{\ln^2 n} \int_0^{1/n} |\ln t| dt \right) \geq \sum_{n=10}^{\infty} \theta \left(\frac{1}{n \ln n} \right) \geq \sum_{n=10}^{\infty} \frac{1}{n \ln n} = \infty$$

Lemma

Let a Young function θ satisfy Δ'-condition $j \in \mathbb{Z}$, φ and $\widetilde{\varphi}$ be compactly supported functions, $\varphi \in L_\theta$, $\widetilde{\varphi} \in L_\infty$. Then for every $f \in L_\theta$

$$I_\theta\left(\sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk}\right) \le C_{\varphi,\widetilde{\varphi}} I_\theta(f), \tag{1}$$

Lemma

Let $f, \varphi \in L_\theta$, $\widetilde{\varphi} \in L_\infty$ be compactly supported refinable functions. Suppose $\{\psi_{jk}^{(l)}\}_{j,k,l}$, $\{\widetilde{\psi}_{jk}^{(l)}\}_{j,k,l}$ are dual wavelet systems generated from φ, $\widetilde{\varphi}$ by MEP. Then

$$\sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{0k}\rangle \varphi_{0k} + \sum_{i=0}^{j-1} \sum_{l=1}^{r} \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\psi}_{ik}^{(l)}\rangle \psi_{ik}^{(l)} = \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk}$$

Theorem

Let θ satisfy Δ'-condition, $s \in \mathbb{N}$, $f \in W^s_\theta$. Suppose $\varphi \in L_\theta$, $\widetilde{\varphi} \in L_\infty$ are refinable compactly supported functions and $\{\psi^{(l)}_{jk}\}_{j,k,l}$, $\{\widetilde{\psi}^{(l)}_{jk}\}_{j,k,l}$ are dual wavelet systems generated from φ, $\widetilde{\varphi}$ by MEP and such that the system $\{\psi^{(l)}_{ik}\}_{i,k,l}$ has VM^s property.
Then

$$\left\| f - \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk}\rangle \varphi_{jk} \right\|_\theta \le C\rho^{-sj} \|f\|_{W^s_\theta},$$

where $\rho > 1$ is any number that is less than any eigenvalue of M in absolute value and C does not depend on f and j, i.e., the wavelet expansion has approximation order s in the sense of convergence in the Luxemburg norm.

Thank you very much!