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On smoothness of generalized eigenfunctions for differential-difference operators

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НАЦИОНАЛЬНЫЕ ПРОЕКТЫ РОССИИ

НАУКА И УНИВЕРСИТЕТЫ

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Generalized solutions of boundary value problems for differential–difference equations on a finite interval were first considered in [1], [2]. It was shown that the smoothness of generalized solutions can be violated at interior points of the interval even for an infinitely differentiable right-hand side of the equation.

Boundary value problems for functional differential equations appear in control theory and, in particular, in the problem of damping a control system with aftereffect [3]–[5]. In [6]–[10], conditions on the right-hand sides of the differential–difference equations were obtained, which guarantee the existence of generalized solutions preserving the smoothness on the entire interval.

There also arises a question: "Under what conditions on the coefficients of a difference operator the smoothness of generalized solutions of boundary value problems for differential–difference equations is preserved on the entire interval for any right-hand side?" The papers [11] and [12] deal with the study of this issue.

However, there arises another unsolved problem: "Will the generalized eigenfunctions of differential–difference operators preserve their smoothness on the entire interval or not?" The present paper is devoted to the study of this problem.

Let us define a difference operator $R : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ by the formula

$$(Ru)(x) = \sum_{j=-n}^{n} \alpha_j u(x+j), \tag{1}$$

where $\alpha_j \in \mathbb{C}$.
Let Q be an interval $(0, d)$, where $d = n + \Theta, n \in \mathbb{N}, 0 < \Theta \leq 1$.
The shifts of arguments $x \to x + j$ in the operator R can map the points of the interval Q into $\mathbb{R} \setminus Q$.
We also introduce an operator $R_Q : L_2(Q) \to L_2(Q)$ by the formula

$$R_Q = P_Q R I_Q,$$

where $I_Q : L_2(Q) \to L_2(\mathbb{R})$ is the operator of extension by zero of a function from $L_2(Q)$ to $\mathbb{R} \setminus Q$, $P_Q : L_2(\mathbb{R}) \to L_2(Q)$ is the operator of the restriction of a function from $L_2(\mathbb{R})$ to Q.

Consider the partition of the interval $Q = (0, d)$ into subintervals formed from this interval by deleting the orbits of its endpoints under the group of integer shifts.
If $\Theta = 1$, then we obtain one class of disjoint subintervals $Q_{1k} = (k-1, k)$ for $k = 1, ..., n+1$;
if $0 < \Theta < 1$, then we have two classes of disjoint intervals $Q_{1k} = (k-1, k-1+\Theta)$, $k = 1, ..., n+1$, and $Q_{2k} = (k-1+\Theta, k)$, $k = 1, ..., n$.
Define the vector function $(U_s u)(x) := (u_1^s, ..., u_N^s)^T$ by the formula

$$u_k^s(x) = u(t+k-1), \quad x \in Q_{s1}, \quad k = 1, ..., N, \tag{2}$$

where $N = n+1$ for $s = 1$, $\quad N = n$ for $s = 2$.

Denote by R_1 the $(n+1)$ x $(n+1)$ matrix that has the form

$$R_1 = \begin{pmatrix} \alpha_0 & \alpha_1 & \dots & \alpha_n \\ \alpha_{-1} & \alpha_0 & \dots & \alpha_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{-n} & \alpha_{-n+1} & \dots & \alpha_0 \end{pmatrix},$$

and R_2 - the n x n matrix of the form

$$R_2 = \begin{pmatrix} \alpha_0 & \alpha_1 & \dots & \alpha_{n-1} \\ \alpha_{-1} & \alpha_0 & \dots & \alpha_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{-n+1} & \alpha_{-n+2} & \dots & \alpha_0 \end{pmatrix}.$$

It is clear that the matrix R_2 can be obtained from R_1 by deleting the last row and the last column.

Lemma 1. The operator $R_{Q_s} = U_s R_Q U_s^{-1} : L_2^N(Q_{s1}) \to L_2^N(Q_{s1})$ is the operator of multiplication by the square matrix R_s.

We introduce a differential–difference operator
$A_R : L_2(0,d) \supset D(A_R) \to L_2(0,d)$ by the formula

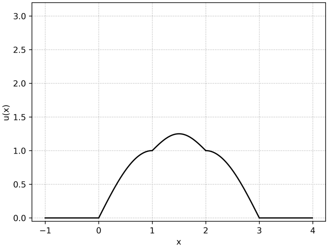
$$A_R u = -\frac{d^2 R_Q u}{dx^2}, \quad u \in D(A_R) = \{u \in \mathring{W}_2^1(Q) : R_Q u \in W_2^2(Q)\}. \quad (3)$$

Definition 1. A function $0 \neq u \in D(A_R)$ is called a generalized eigenfunction of the operator A_R, corresponding to an eigenvalue λ if

$$A_R u = \lambda u. \quad (4)$$

Example 1. We define the difference operator $R : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ by the formula

$$Ru(x) = 2u(x) + u(x-1) + u(x+1), \quad Q = (0,3), \quad \Theta = 1.$$



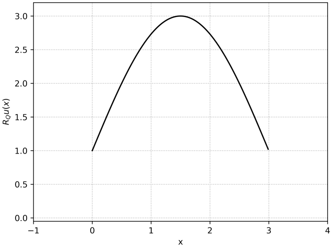


Fig. 1. $u(x)$

Fig. 2. $R_Q u(x)$

Definition 2. We say that the differential–difference operator $-\frac{d^2R_Q}{dx^2}$ satisfies the strong ellipticity condition if the matrix $R_1+R_1^*$ is positive definite, where R_1^* is the Hermitian adjoint matrix.

Theorem 1. Let the operator A_R be strongly elliptic. Then the spectrum $\sigma(A_R)$ is discrete, and $\sigma(A_R)\subset\{\lambda\in\mathbb{C}: Re\lambda>0\}$.

If, moreover, $\alpha_j=\overline{\alpha}_{-j}\quad(|j|\le n)$, then the operator A_R is self-adjoint, and $\sigma(A_R)\subset\mathbb{R}_+=\{\lambda\in\mathbb{R}:\lambda>0\}$.

Theorem 2. Let $detR_s\neq 0\quad(s=1,2$ for $0<\Theta<1;\quad s=1$ for $\Theta=1)$, and let $0\neq u$ be a generalized eigenfunction of the operator A_R corresponding to the eigenvalue λ.

Then $u\in W_2^2(j-1,\ j),\quad j=1,...,n+1$, if $\Theta=1$, and

$u\in W_2^2(j-1,j-1+\Theta),\quad j=1,...,n+1,$

$u\in W_2^2(j-1+\Theta,j),\quad j=1,...,n$, if $0<\Theta<1$.

Remark 1. If the operator A_R is strongly elliptic, then $detR_s \neq 0$ $(s = 1, 2 \text{ for } 0 < \Theta < 1; \quad s = 1 \text{ for } \Theta = 1)$. Thus, the conclusion of Theorem 2 on the smoothness of generalized eigenfunctions on the subintervals Q_{sj} holds for generalized eigenfunctions of the strongly elliptic operator A_R.
However, there arises a question: "Is the smoothness of generalized eigenfunctions preserved on the entire interval $(0, d)$?"
To answer this question, let us reduce problem (4) to a system of ordinary differential equations with spectral parameter λ and with nonlocal boundary conditions.

We restrict ourselves to the case $\Theta = 1$.
Let $0 \neq u \in D(A_R)$ be a generalized eigenfunction of A_R corresponding to the eigenvalue λ. Then equality (4) can be rewritten as

$$-V''(x) = \lambda R_1^{-1} V(x), \quad x \in (0,1), \tag{5}$$

where $V(x) = (U_1 u)(x)$ satisfies the conditions:

$$v_1(0) = 0, \tag{6}$$

$$v_{n+1}(1) = 0, \tag{7}$$

$$v_k(1) = v_{k+1}(0), \quad k = 1, ..., n, \tag{8}$$

$$(R_1 V')_k(1) = (R_1 V')_{k+1}(0), \quad k = 1, ..., n. \tag{9}$$

Lemma 2. The general solution of the system of ordinary differential equations (5) has the form

$$V(x) = e^{i\sqrt{\lambda}\sqrt{R_1^{-1}}x} C_o + e^{-i\sqrt{\lambda}\sqrt{R_1^{-1}}x} C_e, \tag{10}$$

where the $(n+1)$ - vectors $C_o = (C_1, C_3, ..., C_{2n+1})^T$, $C_e = (C_2, C_4, ..., C_{2n+2})^T \in \mathbb{C}^{n+1}$ are arbitrary.
Substituting (10) into the boundary conditions (6) - (9), we obtain the system of linear equations

$$A(\lambda)C = 0, \tag{11}$$

where $C = (C_1, C_2, ..., C_{2n+1}, C_{2n+2})^T \neq 0$.
A necessary and sufficient condition for system (11) to have a nontrivial solution is that $detA(\lambda) = 0$. Thus, the set of eigenvalues of the operator A_R coincides with the set of roots of the determinant $detA(\lambda)$.

If we additionally require that the smoothness of generalized eigenfunctions is preserved over the entire interval, i.e. $u \in W_2^2(0, n+1)$, then we must add conditions (6)-(9) with the additional conditions

$$v_k'(1) = v_{k+1}'(0), \quad k = 1, ..., n. \tag{12}$$

Then the general solution of system (5) in the form (10) must be substituted into the $3n+2$ the boundary conditions (6)-(9), (12). We obtain a system

$$B(\lambda)C = 0 \tag{13}$$

of $3n+2$ equations for $2n+2$ unknowns.

Theorem 3. Let the operator A_R be strongly elliptic. Further, assume that $\Theta = 1$ and $detA(\lambda) = 0$. Then there exists a generalized eigenfunction $u \in \mathring{W}_2^1(0,d) \setminus W_2^2(0,d)$ corresponding to the eigenvalue λ if and only if $rangB(\lambda) > rangA(\lambda)$.

Example 2. We define the difference operator $R : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ by the formula

$$(Ru)(x) = u(x) + \alpha u(x-1) + \alpha u(x+1), \quad \alpha \in \mathbb{R}, \quad x \in \mathbb{R}. \qquad (14)$$

Let $Q = (0, 2)$. We introduce an operator A_R by formula (3). The matrix R_1 has the form

$$R_1 = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}.$$

We assume that α is an irrational number such that $|\alpha| < 1$. Then the matrix R_1 is positive definite. Thus, the operator A_R is strongly elliptic, and, by Theorem 1, it is also self-adjoint, and one has $\sigma(A_R) \subset \mathbb{R}_+$. Consider the eigenfunction–eigenvalue problem

$$A_R u = \lambda u \qquad (15)$$

for the operator A_R.

Let $v_k(x) = u(x+k-1), \quad x \in (0,1), \quad k = 1,2,$ then the generalized eigenfunctions have the form

$$v_1 = C_1 e^{\sqrt{\frac{-\lambda}{1+\alpha}}x} + C_2 e^{-\sqrt{\frac{-\lambda}{1+\alpha}}x} + C_3 e^{\sqrt{\frac{-\lambda}{1-\alpha}}x} + C_4 e^{-\sqrt{\frac{-\lambda}{1-\alpha}}x}, \tag{16}$$

$$v_2 = C_1 e^{\sqrt{\frac{-\lambda}{1+\alpha}}x} + C_2 e^{-\sqrt{\frac{-\lambda}{1+\alpha}}x} - C_3 e^{\sqrt{\frac{-\lambda}{1-\alpha}}x} - C_4 e^{-\sqrt{\frac{-\lambda}{1-\alpha}}x}. \tag{17}$$

Substituting (16) and (17) into the boundary conditions and performing some transformations, we obtain

$$C_1 + C_2 + C_3 + C_4 = 0, \tag{18}$$

$$C_1\left(e^{\sqrt{\frac{-\lambda}{1+\alpha}}} - 1\right) + C_2\left(e^{-\sqrt{\frac{-\lambda}{1+\alpha}}} - 1\right) = 0, \tag{19}$$

$$C_3\left(e^{\sqrt{\frac{-\lambda}{1-\alpha}}} + 1\right) + C_4\left(e^{-\sqrt{\frac{-\lambda}{1-\alpha}}} + 1\right) = 0, \tag{20}$$

$$C_2\sqrt{\lambda(1+\alpha)}\left(-e^{-\sqrt{\frac{-\lambda}{1+\alpha}}} + 1\right) + C_3\sqrt{\lambda(1-\alpha)}\left(e^{\sqrt{\frac{-\lambda}{1-\alpha}}} + 1\right) = 0. \tag{21}$$

The determinant of system (18)-(21) is equal to zero in the following three cases:

$$1. \quad e^{\sqrt{\frac{-\lambda}{1-\alpha}}} + 1 = 0. \tag{22}$$

Then we have $\lambda_k = (1-\alpha)(\pi + 2\pi k)^2$,

$$u_k(x) = \sin(\pi(1+2k)x), \quad x \in (0,2), \quad k = 0, 1, 2, \ldots.$$

$$2. \quad e^{\sqrt{\frac{-\lambda}{1+\alpha}}} - 1 = 0. \tag{23}$$

Then we obtain $\lambda_k = (1+\alpha)4\pi^2 k^2$,

$$u_k(x) = \sin(2\pi kx), \quad x \in (0,2), \quad k = 1, 2, \ldots.$$

$$3. \quad 1 - \sqrt{\frac{1+\alpha}{1-\alpha}}\,\mathrm{tg}\Big(\frac{1}{2}\sqrt{\frac{\lambda}{1-\alpha}}\Big)\,\mathrm{tg}\Big(\frac{1}{2}\sqrt{\frac{\lambda}{1+\alpha}}\Big) = 0. \tag{24}$$

Note that equations (22), (23), and (24) have countably many roots.

Let us add additional condition that provides preservation of smoothness of eigenfunctions at the point $x = 1$ (see (12)). This condition has the form

$$C_1\sqrt{\frac{\lambda}{1+\alpha}}(e^{\sqrt{\frac{-\lambda}{1+\alpha}}}-1)+C_2\sqrt{\frac{\lambda}{1+\alpha}}(-e^{-\sqrt{\frac{-\lambda}{1+\alpha}}}+1)+C_3\sqrt{\frac{\lambda}{1-\alpha}}\times$$

$$(e^{\sqrt{\frac{-\lambda}{1-\alpha}}}+1)+C_4\sqrt{\frac{\lambda}{1-\alpha}}(-e^{-\sqrt{\frac{-\lambda}{1-\alpha}}}-1)=0. \qquad (25)$$

Now let us verify that the conditions of Theorem 3 are satisfied. To this end, we consider the determinant $\Delta_1(\lambda)$ of the system formed by Eqs. (18)–(20), and (25), and show that if λ is a root of (24), then $\Delta_1 \neq 0$, and the rank of system (18)–(21), (25) is equal to 4.

The determinant of the system (18) - (20), (25) turns to zero in the following cases:

$$1. \quad e^{\sqrt{\frac{-\lambda}{1-\alpha}}} + 1 = 0, \tag{26}$$

$$2. \quad e^{\sqrt{\frac{-\lambda}{1+\alpha}}} - 1 = 0, \tag{27}$$

$$3. \quad 1 - \sqrt{\frac{1-\alpha}{1+\alpha}}\,\mathrm{tg}\Big(\frac{1}{2}\sqrt{\frac{\lambda}{1-\alpha}}\Big)\,\mathrm{tg}\Big(\frac{1}{2}\sqrt{\frac{\lambda}{1+\alpha}}\Big) = 0. \tag{28}$$

It is possible to prove that for irrational number $\alpha \neq 0$ equations (24) and (26), (24) and (27), (24) and (28) have no common roots. Therefore, if Eq.(24) is satisfied, then $\Delta_1 \neq 0$, and the rank of system (18)–(21), (25) is equal to 4. Then it follows from Theorem 3 that the eigenvalues λ of the operator A_R, which are also the roots of Eq.(24), correspond to the generalized eigenfunctions $u \in \mathring{W}_2^1(0,d) \setminus W_2^2(0,d)$.

Thus, we have shown that the differential–difference operator A_R generated by the difference operator R given by formula (14) has countably many generalized eigenfunctions whose smoothness is violated inside the interval and countably many generalized eigenfunctions whose smoothness is preserved.

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English translation in: Math. Notes, 114:5 (2023), 1002–1020.

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