Formation of singularities of two-dimensional soliton equations represented by L, A, B-triples

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Seminar on Analysis, Differential Equations, and Mathematical Physics
Southern Federal University, Rostov-na-Donu, May 18, 2023

Lax pairs

The Korteweg–de Vries (KdV) equation

$$u_t = 6uu_x + u_{xxx}$$

and other classical 1D soliton equations integrated by the inverse scattering method are represented by Lax pairs

$$L_t = [L, A],$$

where L and A are differential operators. For the KdV equation we have

$$L = -\frac{d^2}{dx^2} + u(x, t).$$

These evolutions preserve the spectrum of the operator L deforming the eigenfunctions as

$$\psi_t + A\psi = 0, \quad \text{where } L\psi = E\psi.$$

The Darboux transformation

Every solution ω of the equation $L\omega = 0$ defines a factorization of L:

$$L = A^\top A, \quad A = -\frac{d}{dx} + v, \quad A^\top = \frac{d}{dx} + v, \quad v = \frac{\omega'}{\omega}.$$

The Darboux transformation of H consists in swapping $A^\top$ and A:

$$L = A^\top A \;\longrightarrow\; \widetilde{L} = AA^\top = -\frac{d^2}{dx^2} + \widetilde{u}(x),$$

and it acts on eigenfunctions as follows:

$$\psi \longrightarrow \widetilde{\psi} = A\psi.$$

The Darboux transformation is extended to solutions of the KdV equation.

L, A, B-triples

L, A, B-triples were introduced by Manakov in the middle of 1970s as 2D equations presented in the form

$$L_t = [L, A] + BL$$

where L, A, and B are partial differential operators.
The spectrum is not preserved except its zero level

$$L\psi = 0$$

which evolves as in the case of Lax pairs:

$$\psi_t + A\psi = 0.$$

The Novikov–Veselov (NV) equation

$$U_t = U_{zzz} + U_{\bar{z}\bar{z}\bar{z}} + 3(VU)_z + 3(\bar{V}U)_{\bar{z}} = 0,$$

$$V_{\bar{z}} = U_z,$$

with

$$L = \partial\bar{\partial} + U,$$

$U = U(z, \bar{z}, t), U = \bar{U}.$

In one-dimensional limit, $U = U(x)$, it reduces to the KdV equation.

This equation was derived in the framework of two-dimensional Schrodinger operators which are finite-gap on one level of energy. Such operatos were introduced in 1976 by Dubrovin, Krichever, and Novikov.

The modified Novikov–Veselov (mNV) equation

$$U_t = \left(U_{zzz} + 3U_z V + \frac{3}{2} U V_z\right) + \left(U_{\bar{z}\bar{z}\bar{z}} + 3U_{\bar{z}} \bar{V} + \frac{3}{2} U \bar{V}_{\bar{z}}\right),$$

$$V_{\bar{z}} = (U^2)_z$$

where

$$L = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix},$$

$U = \bar{U}$.

This equation was introduced in the early 1990s by Bogdanov. In one-dimensional limit, $U = U(x)$, it reduces to the modified Korteweg–de Vries equation.

The (focusing) Davey–Stewartson II (DS II) equation

$$U_t = i(U_{zz} + U_{\bar{z}\bar{z}} + (V + \bar{V})U),$$
$$V_{\bar{z}} = 2(|U|^2)_z,$$

where

$$L = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & \bar{U} \end{pmatrix}.$$

The Moutard transformation

Let $L = \partial\bar{\partial} + u$ and

$$L\omega = (-\Delta + u)\omega = 0,$$

where Δ is the two-dimensional Laplace operator:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

The Moutard transformation of L is defined as

$$\widetilde{L} = -\Delta + u - 2\Delta \log \omega = -\Delta - u + 2\frac{\omega_x^2 + \omega_y^2}{\omega^2}.$$

If ψ satisfies $L\psi = 0$, then the function θ, defined via the system

$$(\omega\theta)_x = -\omega^2 \left(\frac{\psi}{\omega}\right)_y, \quad (\omega\theta)_y = \omega^2 \left(\frac{\psi}{\omega}\right)_x,$$

satisfies $\widetilde{L}\theta = 0$.

The Moutard transformation and formation of singularities

To all these three equations (NV, mNV, and DSII) by using the Moutard type transformations were constructed solutions such that
1) their initial data are smooth and fast decaying,
2) solutions are rational,
3) at certain critical time these solutions become singular.

- Novikov-Veselov (2008, T.–Tsarev, classical Moutard transformation, the two-dimensional generalization of the Darboux transformation for operators $L = \partial\bar{\partial} + U$),
- modified Novikov-Veselov (2016, T.),
- Davey–Stewartson II (2021, T.).

In all these cases the zero level energy spectrum degenerates as t approaches the terminal time.

- Given
$$F(x, y, t) = 2(x^4 + y^4) + \frac{8}{3}(x^3 + y^3) + 4x^2y^2 + 20 - 8t,$$
for
$$t < T_{sing} = \frac{29}{12};$$
the functions
$$U = 2\partial\bar{\partial} \log F, \quad V = 2\partial^2 \log F$$
satisfy the Novikov–Veselov equation and
$$L\psi = (\partial\bar{\partial} + U)\psi = 0, \quad \psi = \frac{xy}{F};$$

- Given

$$a = i(x^2 - y^2), \;\; b = -y\left(\frac{y^2}{3} - x^2 - 1\right) - i\left(x\left(1 + y^2 - \frac{x^2}{3}\right) + C - t\right)$$

for

$$t \neq T_{sing} = C$$

the function

$$U = i\frac{a(|z|^2 - 1) + \bar{b}\bar{z} - bz}{|a|^2 + |b|^2}$$

satisfies the modified Novikov–Veselov equation and

$$L\psi = \left[\begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}\right]\psi = 0$$

where

$$\psi = \frac{1}{|a|^2 + |b|^2}\begin{pmatrix} \bar{a}z - \bar{b} \\ \bar{a} + \bar{b}\bar{z} \end{pmatrix}$$

and the polynomial $|a|^2 + |b|^2$ vanishes if and only if $C = t$.
Moreover

$$\int_{\mathbb{R}^2} U^2\, dxdy = \begin{cases} 3\pi & \text{for } t \neq C, \\ 2\pi & \text{for } t = C. \end{cases}$$

- For

$$t \neq T_{sing} \neq C \in \mathbb{R}$$

the function

$$U = i\frac{z^2 - 2i(t-C)}{|z|^2 + |z^2 + 2i(t-C)|^2}$$

satisfies the Davey–Stewartson II equation and

$$L\psi = \left[\begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & \bar{U} \end{pmatrix}\right]\psi = 0$$

where

$$\psi = \frac{1}{|z|^2 + |z^2 + 2i(t-C)|^2}\begin{pmatrix} \bar{z} \\ -iz^2 + 2(t-C) \end{pmatrix}.$$

Moreover

$$\int_{\mathbb{R}^2} |U|^2\, dxdy = \begin{cases} 2\pi & \text{for } t \neq C, \\ \pi & \text{for } t = C. \end{cases}$$

The Weierstrass representation of surfaces in $\mathbb{R}^4$.I

$$\mathcal{D} = L = \begin{pmatrix} 0 & \partial \\ -\bar\partial & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & \bar U \end{pmatrix}, \quad \mathcal{D}^\vee = \begin{pmatrix} 0 & \partial \\ -\bar\partial & 0 \end{pmatrix} + \begin{pmatrix} \bar U & 0 \\ 0 & U \end{pmatrix}$$

$$\mathcal{D}\psi = 0, \qquad \mathcal{D}^\vee \varphi = 0.$$

then the formulae

$$x^k(P) = x^k(P_0) + \int \left(x^k_z dz + \bar x^k_z d\bar z \right), \quad k = 1, \dots, 4,$$

$$x^1_z = \frac{i}{2}(\bar\varphi_2 \bar\psi_2 + \varphi_1 \psi_1), \qquad x^2_z = \frac{1}{2}(\bar\varphi_2 \bar\psi_2 - \varphi_1 \psi_1),$$

$$x^3_z = \frac{1}{2}(\bar\varphi_2 \psi_1 + \varphi_1 \bar\psi_2), \qquad x^4_z = \frac{i}{2}(\bar\varphi_2 \psi_1 - \varphi_1 \bar\psi_2),$$

define the surface in $\mathbb{R}^4$ (Konopelchenko)

The Weierstrass representation of surfaces in $\mathbb{R}^4$.II

If $z = u + iv$ is a conformal parameter, i.e.

$$ds^2 = e^{\alpha}(du^2 + dv^2),$$

then

$$\left(\frac{\partial X}{\partial u}, \frac{\partial X}{\partial u}\right) = \left(\frac{\partial X}{\partial v}, \frac{\partial X}{\partial v}\right),$$

$$\left(\frac{\partial X}{\partial u}, \frac{\partial X}{\partial v}\right) = 0,$$

and

$$\sum \left(\frac{\partial X^k}{\partial z}\right)^2 = \left[\left(\frac{\partial X}{\partial u}, \frac{\partial X}{\partial u}\right) - \left(\frac{\partial X}{\partial v}, \frac{\partial X}{\partial v}\right)\right] +$$

$$+2i\left(\frac{\partial X}{\partial u}, \frac{\partial X}{\partial v}\right),$$

i.e.

$$\sum \left(\frac{\partial X^k}{\partial z}\right)^2 = 0.$$

The Weierstrass representation of surfaces in $\mathbb{R}^4$.III

$\widetilde{G}_{n,2}$ is the quadric Q:

$$z_1^2 + \cdots + z_n^2 = 0, \quad (z_1 : \cdots : z_n) \in Q_n \subset \mathbb{C}P^{n-1}.$$

For a surface with a conformal parameter z, we define the Gauss map as

$$z \to \left(\frac{\partial X^1}{\partial z} : \cdots : \frac{\partial X^n}{\partial z} \right) \in Q_n.$$

For $n = 4$ we have the diffeomorphic Segre mapping

$$\mathbb{C}P^1 \times \mathbb{C}P^1 \to Q_4$$

$$z_1 = \frac{i}{2}(a_1 b_1 + a_2 b_2), \; z_2 = \frac{1}{2}(a_2 b_2 - a_1 b_1),$$

$$z_3 = \frac{1}{2}(a_1 b_2 - a_2 b_1), z_4 = \frac{i}{2}(a_2 b_1 - a_1 b_2),$$

where $(a_1 : a_2) \in \mathbb{C}P^1, (b_1 : b_2) \in \mathbb{C}P^1$.

The Weierstrass representation of surfaces in $\mathbb{R}^4$.IV

The spinors ψ and φ take the form

$$\varphi = (a_1, \bar{a}_2), \quad \psi = (b_1, \bar{b}_2)$$

and are reconstructed up to the gauge transformations

$$\psi_1 \to e^h \psi_1, \; \psi_2 \to e^{\bar{h}} \psi_2,$$

$$\varphi_1 \to e^{-h} \varphi_1, \; \varphi_2 \to e^{-\bar{h}} \varphi_2, \; U \to e^{\bar{h}-h} U,$$

with h holomorphic.

Every surface in $\mathbb{R}^4$ has a global Weierstrass (spinor) representation (T., 2006).

Q_3 is $\mathbb{C}P^1$ and for $n > 4$ the quadrics Q_n have no such rational parameterizations.

DSII deformation of surfaces

Since $\mathcal{D}$ enters the L, A, B-triples for the DS hierarchy, Konopelchenko (2000) proposed to construct soliton deformations of surfaces:

$$\psi_t = A\psi, \quad \varphi_t = A^\vee \varphi$$

give a deformation of the Gauss map and it defines a deformation of surfaces up to translations by $r(t) \in \mathbb{R}^4$.
Such deformations also depend on resolving the constraint

$$V_{\bar{z}} = 2(|U|^2)_z.$$

One can define a deformation governed by DSII globally such that it preserves

$$\int |U|^2 dx \wedge dy$$

which is (up to a multiple) the Willmore functional of the surface (T., 2006).

The Moutard transformation of solutions of DSII.I

Let us write down the surface in the form

$$S(\varphi,\psi)(z,\bar{z}) =$$

$$= \int \left[i \begin{pmatrix} \psi_1\bar{\varphi}_2 & -\bar{\psi}_2\bar{\varphi}_2 \\ \psi_1\varphi_1 & -\bar{\psi}_2\varphi_1 \end{pmatrix} dz + i \begin{pmatrix} \psi_2\bar{\varphi}_1 & \bar{\psi}_1\bar{\varphi}_1 \\ -\psi_2\varphi_2 & -\bar{\psi}_1\varphi_2 \end{pmatrix} d\bar{z} \right] =$$

$$= \int d \begin{pmatrix} ix^3 + x^4 & -x^1 - ix^2 \\ x^1 - ix^2 & -ix^3 + x^4 \end{pmatrix}$$

with $S \in \mathbb{H} = \mathbb{R}^4$.
Then take its DSII deformation $S(z,\bar{z},t)$ and for every moment of time take its Moebius transform:

$$S \to S^{-1}.$$

The Moutard transformation of solutions of DSII.II

Given a surface we a conformal parameter z, we define its potential U.
The Moebius transformation is conformal and hence we get a transformation of potentials

$$U(z, \bar{z}, t) \rightarrow \widetilde{U}(z, \bar{z}, t)$$

which maps solutions of DSII into solutions of DSII (T., 2021).

Minimal surfaces in $\mathbb{R}^4$ and solutions to DSII

Let $f(z,t)$ be a function which is holomorphic in z and satisfies the equation

$$\frac{\partial f}{\partial t} = i\frac{\partial^2 f}{\partial z^2}.$$

It defines the DSII deformation of the minimal surface

$$z_2 = f(z_1) \quad \text{in } \mathbb{C}^2 = \mathbb{R}^4.$$

Then

$$U = \frac{i(zf' - f)}{|z|^2 + |f|^2}, \quad V = 2ia_z,$$

where

$$a = -\frac{i(\bar{z} + f')\bar{f}}{|z|^2 + |f|^2},$$

satisfy the Davey–Stewartson II equation (T., 2021).

Example

$$f = z^4 + 12itz^2 - 12t^2 + c,$$

$$U = \frac{i(3z^4 + 12itz^2 + 12t^2 - c)}{|z|^2 + |z^4 + 12itz^2 - 12t^2 + c|^2}$$

becomes singular for $c = 12t^2$ and it has singularities

$$U \sim -12te^{2i\phi} \quad \text{at } z = 0 \text{ for } t = \pm\sqrt{c/12}.$$

The first integral $\int |U|^2 dxdy$ is equal to 4π for t such that U is nonsingular and is equal to 3π for $t = T_{\mathrm{sing}}$.

The multiplicity of the value of this functional to π in both cases is explained by that the surfaces S^{-1} are immersed Willmore spheres (with singularities for singular moments of time). First time, such an effect was established for the mNV equation (T., 2016).

The Ozawa solution

$$U(x,y,0) = \frac{e^{-i(b/a)(y^2-x^2)}}{a + 2(x^2+y^2)/a}$$

Let

$$-\frac{a}{b} = T_{sing} > 0,$$

then

$$\int |U|^2 dxdy \to 2\pi\delta$$

for $t \to T_{sing}$, where δ is the Dirac distribution centered at the origin, and at another moments of time the solution is nonsingular.