Perturbations of Periodic Sturm-Liouville Operators

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What is a periodic Sturm-Liouville operator ?

Here is a periodic Sturm-Liouville differential expression ℓ:

$$\ell_0(f) = \frac{1}{r_0}\big(-(p_0 f')' + q_0 f\big) \quad \text{on } \mathbb{R},$$

$r_0, 1/p_0, q_0 \in L^1_{\text{loc}}(\mathbb{R})$ real functions, periodic and $r_0, p_0 > 0$ a.e.

Applications

- Schrödinger operators for atomic nuclei in a crystal.
- Perturbations well studied for $r_0 = p_0 = 1$. Some is known for $r_0 = r_1$. We mention here only: F. Gesztesy, F.S. Rofe-Beketov, K.-M. Schmidt, B. Simon, G. Teschl, and J. Weidmann.
- Some PDE's (Camassa-Holm, Hain-Lüst) lead to weights $r_0 \not> 0$ – this was our initial interest and leads to perturb r_0 (not today).

Unperturbed periodic Sturm-Liouville operator: Spectrum

$$\ell_0(f) = \frac{1}{r_0}\big(-(p_0 f')' + q_0 f\big) \quad \text{on } \mathbb{R},$$

$r_0, 1/p_0, q_0 \in L^1_{\text{loc}}(\mathbb{R})$ real functions, periodic and $r_0, p_0 > 0$ a.e.

Operator

- With ℓ_0 one associates the maximal operator A_0

$$A_0 f := \ell_0(f), \quad f \in \text{dom } A := \mathcal{D}_{max}.$$

- THEN: A_0 is self-adjoint in the weighted L^2-space $L^2(\mathbb{R}; r_0)$.
- Spectrum $\sigma(A_0)$ of A_0 is semibounded from below,
- it is purely absolutely continuous, $\sigma(A_0) = \sigma_{ac}(A_0)$, and
- consists of (finitely or infinitely many) spectral bands.

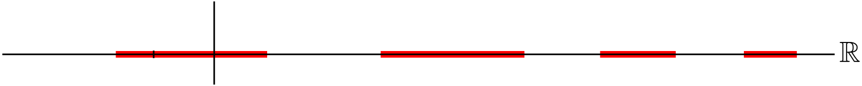
Unperturbed periodic Sturm-Liouville operator: Bands

$$\ell_0(f) = \frac{1}{r_0}\big(-(p_0 f')' + q_0 f\big) \quad \text{on } \mathbb{R},$$

$r_0, 1/p_0, q_0 \in L^1_{\text{loc}}(\mathbb{R})$ real functions, periodic and $r_0, p_0 > 0$ a.e.

Spectrum of A_0

- $\sigma(A_0) = \sigma_{ac}(A_0)$, consists of (finitely/infinitely many) spectral bands:



- The endpoints of the k-th band are the k-th ev's of the regular SL-operator in $L^2((0,\omega); r_0)$ with periodic/semiperiodic boundary conditions.

Recall for a self-adjoint A in a Hilbert space $\mathcal{H}$

- Resolvent set and spectrum of A:

$$\rho(A) := \{\lambda \in \mathbb{C} \mid (A-\lambda)^{-1} \text{ is bd. operator everywhere defined}\}.$$
$$\sigma(A) := \mathbb{C} \setminus \rho(A).$$

- (Normal) Point spectrum:

$$\sigma_p(A) := \{\lambda \in \mathbb{C} \mid A-\lambda \text{ not injective}\}.$$
$$\sigma_{p,norm}(A) := \{\text{isolated ev. with finite dim. kernel}\}.$$

- Essential and absolutely continuous spectrum:

$$\sigma_{ess}(A) := \sigma(A) \setminus \sigma_{p,norm}(A).$$
$$\sigma_{ac}(A) := \sigma(A|\mathcal{H}_{ac}), \text{ where } \mathcal{H}_{ac} := \{x \in \mathcal{H} \mid t \mapsto (E_t x, x) \text{ is absolutely cont. with resp. to Lebesgue}\}.$$

Perturbation I

$$\ell_0(f) = \frac{1}{r_0}\left(-(p_0 f')' + q_0 f\right) \qquad \text{unperturbed.}$$

$r_0, 1/p_0, q_0 \in L^1_{\mathrm{loc}}(\mathbb{R})$ real functions, periodic and $r_0, p_0 > 0$ a.e.

$$\ell_1(f) = \frac{1}{r_1}\left(-(p_1 f')' + q_1 f\right) \qquad \text{perturbed.}$$

$r_1, 1/p_1, q_1 \in L^1_{\mathrm{loc}}(\mathbb{R})$ real functions and $r_1, p_1 > 0$ a.e.

We assume (k=0)

$$\int_{\mathbb{R}} \left(|r_1(t) - r_0(t)| + \left| \frac{1}{p_1(t)} - \frac{1}{p_0(t)} \right| + |q_1(t) - q_0(t)| \right) \mathrm{d}t < \infty$$

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Perturbation II

$$\ell_0(f) = \frac{1}{r_0}\left(-(p_0 f')' + q_0 f\right) \qquad \text{unperturbed.}$$

$r_0, 1/p_0, q_0 \in L^1_{\mathrm{loc}}(\mathbb{R})$ real functions, periodic and $r_0, p_0 > 0$ a.e.

$$\ell_1(f) = \frac{1}{r_1}\left(-(p_1 f')' + q_1 f\right) \qquad \text{perturbed.}$$

$r_1, 1/p_1, q_1 \in L^1_{\mathrm{loc}}(\mathbb{R})$ real functions and $r_1, p_1 > 0$ a.e.

We assume (k=1)

$$\int_{\mathbb{R}} \left(|r_1(t) - r_0(t)| + \left| \frac{1}{p_1(t)} - \frac{1}{p_0(t)} \right| + |q_1(t) - q_0(t)| \right) |t|^1 \, \mathrm{d}t < \infty$$

Perturbation III

$$\ell_0(f) = \frac{1}{r_0}\left(-(p_0 f')' + q_0 f\right) \qquad \text{unperturbed.}$$

$r_0, 1/p_0, q_0 \in L^1_{\mathrm{loc}}(\mathbb{R})$ real functions, periodic and $r_0, p_0 > 0$ a.e.

$$\ell_1(f) = \frac{1}{r_1}\left(-(p_1 f')' + q_1 f\right) \qquad \text{perturbed.}$$

$r_1, 1/p_1, q_1 \in L^1_{\mathrm{loc}}(\mathbb{R})$ real functions and $r_1, p_1 > 0$ a.e.

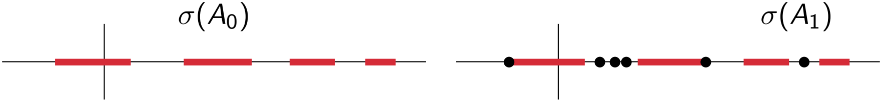
We assume (k=2)

$$\int_{\mathbb{R}} \left(|r_1(t) - r_0(t)| + \left| \frac{1}{p_1(t)} - \frac{1}{p_0(t)} \right| + |q_1(t) - q_0(t)| \right) |t|^2 \, \mathrm{d}t < \infty$$

k=0

We assume (k=0)

$$\int_{\mathbb{R}} \left(|r_1(t) - r_0(t)| + \left| \frac{1}{p_1(t)} - \frac{1}{p_0(t)} \right| + |q_1(t) - q_0(t)| \right) \mathrm{d}t < \infty$$



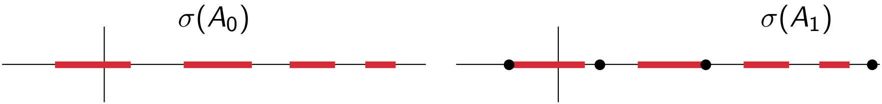
We have

- $\sigma_{ess}(A_1) = \sigma_{ess}(A_0)$
- $\text{int}\,(\sigma_{ess}(A_0)) \subset \sigma_{ac}(A_1)$
- But: Maybe infinitely many eigenvalues in a gap
- But: Band edges can be eigenvalues.

k=1

We assume (k=1)

$$\int_{\mathbb{R}} \left(|r_1(t) - r_0(t)| + \left| \frac{1}{p_1(t)} - \frac{1}{p_0(t)} \right| + |q_1(t) - q_0(t)| \right) |t| \mathrm{d}t < \infty$$

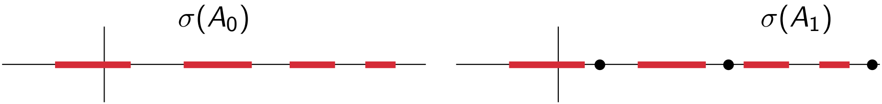


- $\sigma_{ess}(A_1) = \sigma_{ess}(A_0)$
- $\sigma_{ac}(A_1) = \operatorname{int}(\sigma_{ess}(A_0))$
- Now: finitely many eigenvalues in a gap. “Rofe-Beketov type” result.
- But: band edges still can be eigenvalues.

k=2

We assume (k=2)

$$\int_{\mathbb{R}} \left(|r_1(t) - r_0(t)| + \left| \frac{1}{p_1(t)} - \frac{1}{p_0(t)} \right| + |q_1(t) - q_0(t)| \right) |t|^2 \, \mathrm{d}t < \infty$$



- $\sigma_{ess}(A_1) = \sigma_{ess}(A_0)$
- $\sigma_{ac}(A_1) = \sigma_{ess}(A_0)$. Band edges are no ev's.
- Now: finitely many eigenvalues in a gap. "Rofe-Beketov type" result.
- Now: Band edges are no eigenvalues.

Thank You.