Seminar on Analysis, Differential Equations and Mathematical Physics

Schrödinger equation with finitely many δ-interactions: closed form, integral and series representations for solutions

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Introduction

In this talk, we consider the 1D Schrödinger equation of the form

$$-y'' + \left(q(x) + \sum_{k=1}^{N} \alpha_k \delta(x - x_k) \right) y = \lambda y, \quad 0 < x < b, \ \lambda \in \mathbb{C}, \quad (1)$$

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- $q \in L_2(0, b)$ is a complex valued function.
- $\delta(x)$ is the Dirac delta distribution.
- $0 < x_1 < x_2 < \cdots < x_N < b$ and $\alpha_1, \ldots, \alpha_N \in \mathbb{C} \setminus \{0\}$ are the *point interactions*.

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- Another approach consists in considering the interval $[0, b]$ as a quantum graph whose edges are the segments $[x_k, x_{k+1}]$, $k = 0, \ldots, N$, (setting $x_0 = 0$, $x_{N+1} = b$), and the Schrödinger operator with the regular potential q as an unbounded operator on the direct sum $\bigoplus_{k=0}^{N} H^2(x_k, x_{k+1})$, with the domain given by the families $(y_k)_{k=0}^{N}$ that satisfy the condition of continuity $y_k(x_k-) = y_{k+1}(x_k+)$ and the jump condition for the derivative $y'_{k+1}(x_k+) - y'_k(x_k-) = \alpha_k y_k(x_k)$ for $k = 1, \ldots N$:
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- Functional series representations of the transmutation kernel have been constructed and used for practical solving direct and inverse Sturm-Liouville problems

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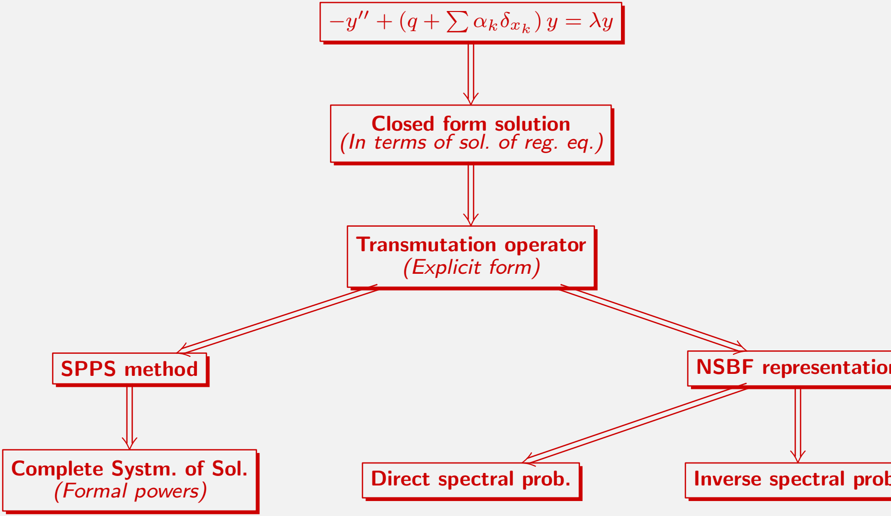

$-y'' + (q + \sum \alpha_k \delta_{x_k})\, y = \lambda y$
Closed form solution
(In terms of sol. of reg. eq.)
Transmutation operator
(Explicit form)
SPPS method
NSBF representatio
Complete Systm. of Sol.
(Formal powers)
Direct spectral prob.
Inverse spectral prob

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- $\mathbf{L}_{q,\mathfrak{I}_N} := \mathbf{L}_q + q_{\delta,\mathfrak{I}_N}(x)$.
- $\mathscr{D}(0,b) = C_0^\infty(0,b)$ (test functions), $\mathscr{D}'(0,b)$ (distributions), $H^k(0,b) = W^{k,2}(0,b)$, $H_0^1(0,b) = W_0^{1,2}(0,b) = \overline{\mathscr{D}(0,b)}^{H^1}$, $H^{-1}(0,b) = (H_0^1(0,b))'$.

- For $u \in L_{2,loc}(0,b)$, $\mathbf{L}_{q,\mathfrak{I}_N}u$ defines a distribution in $\mathscr{D}'(0,b)$ as follows

$$(\mathbf{L}_{q,\mathfrak{I}_N}u,\phi)_{C_0^\infty(0,b)} := \int_0^b u(x)\mathbf{L}_q\phi(x)dx + \sum_{k=1}^N \alpha_k u(x_k)\phi(x_k).$$

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- The function u must be well defined at the points x_k, $k = 1, \dots, N$.
- When $u \in H^1(0,b)$, the distribution $\mathbf{L}_{q,\mathfrak{I}_N}u$ can be extended to a functional in $H^{-1}(0,b)$ as follows

$$(\mathbf{L}_{q,\mathfrak{I}_N}u, v)_{H_0^1(0,b)} := \int_0^b \{u'(x)v'(x) + q(x)u(x)v(x)\}dx + \sum_{k=1}^{N} \alpha_k u(x_k)v(x_k).$$

- $F \in \mathscr{D}'(0,b)$ is L_2-regular if there exists $g \in L_2(0,b)$ such that $(F,\phi)_{C_0^\infty(0,b)} = \int_0^b g\phi$.

Proposition

If $u \in L_{2,loc}(0,b)$, then the distribution $\mathbf{L}_{q,\mathfrak{I}_N} u$ is L_2-regular iff the following conditions hold.

1. For each $k = 0, \dots, N$, $u|_{(x_k, x_{k+1})} \in H^2(x_k, x_{k+1})$.
2. $u \in AC[0,b]$.
3. The discontinuities of the derivative u' are located at the points x_k, $k = 1, \dots, N$, and the jumps are given by

$$u'(x_k+) - u'(x_k-) = \alpha_j u(x_k) \quad \text{for } k = 1, \cdots, N. \tag{2}$$

In such case,

$$(\mathbf{L}_{q,\mathfrak{I}_N} u, \phi)_{C_0^\infty(0,b)} = (\mathbf{L}_q u, \phi)_{C_0^\infty(0,b)} \quad \text{for all } \phi \in C_0^\infty(0,b). \tag{3}$$

Closed form solution

- The L_2-**regularization domain** of $\mathbf{L}_{q,\mathfrak{I}_N}$, denoted by $\mathcal{D}_2\left(\mathbf{L}_{q,\mathfrak{I}_N}\right)$, is the set of all functions $u \in L_{2,loc}(0,b)$ satisfying conditions 1,2 and 3 of the previous proposition.

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- A function $u \in L_{2,loc}(0,b)$ is a solution of Eq. (1) iff $u \in \mathcal{D}_2\left(\mathbf{L}_{q,\mathfrak{I}_N}\right)$ and for each $k = 0, \ldots, N$, the restriction $u|_{(x_k,x_{k+1})}$ is a solution of the regular Schrödinger equation

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- In what follows, denote $\lambda = \rho^2$, $\rho \in \mathbb{C}$.
- Let $\widehat{s}_k(\rho,x)$ be the unique solution of the Cauchy problem

$$\begin{cases} -\widehat{s}_k''(\rho,x) + q(x+x_k)\widehat{s}_k(\rho,x) = \rho^2 \widehat{s}_k(\rho,x), & 0 < x < b - x_k, \\ \widehat{s}_k(\rho,0) = 0, \ \widehat{s}_k'(\rho,0) = 1. \end{cases} \tag{5}$$

- $\widehat{s}_k(\rho, x - x_k)$ is the solution of $\mathbf{L}_q u = \rho^2 u$ on (x_k, b) with initial conditions $u(x_k) = 0$, $u'(x_k) = 1$.

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- $(\mathbf{L}_q - \rho^2)\,(H(x - x_k)\widehat{s}_k(\rho, x - x_k)) = -\delta(x - x_k)$ for $x_k < x < b$[1]

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- We denote by $\mathcal{J}_N$ the set of finite sequences $J = (j_1, \dots, j_l)$ with $1 < l \leqslant N$, $\{j_1, \dots, j_l\} \subset \{1, \dots, N\}$ and $j_1 < \cdots < j_l$.

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- Given $J \in \mathcal{J}_N$, the length of J is denoted by $|J|$ and we define $\alpha_J := \alpha_{j_1} \cdots \alpha_{j_{|J|}}$.

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- Given $J \in \mathcal{J}_N$, the length of J is denoted by $|J|$ and we define $\alpha_J := \alpha_{j_1} \cdots \alpha_{j_{|J|}}$.
- χ_A denotes the characteristic function of the interval $[-A, A]$.

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Theorem

Given $u_0, u_1 \in \mathbb{C}$*, the unique solution* $u_{\mathfrak{I}_N} \in \mathcal{D}_2\left(\mathbf{L}_{q,\mathfrak{I}_N}\right)$ *of the Cauchy problem*

$$\begin{cases} \mathbf{L}_{q,\mathfrak{I}_N} u(x) = \lambda u(x), \quad 0 < x < b, \\ u(0) = u_0, \ u'(0) = u_1. \end{cases}$$

has the form

$$u_{\mathfrak{I}_N}(\rho, x) = \widetilde{u}(\rho, x) + \sum_{k=1}^{N} \alpha_k \widetilde{u}(\rho, x_k) H(x - x_k) \widehat{s}_k(\rho, x - x_k)$$

$$+ \sum_{J \in \mathcal{J}_N} \alpha_J H(x - x_{j_{|J|}}) \widetilde{u}(\rho, x_{j_1}) \left(\prod_{l=1}^{|J|-1} \widehat{s}_{j_l}(\rho, x_{j_{l+1}} - x_{j_l}) \right) \widehat{s}_{j_{|J|}}(\rho, x - x_{j_{|J|}})$$

where $\widetilde{u}(\rho, x)$ is the unique solution of the regular Schrödinger equation

$$\mathbf{L}_q \widetilde{u}(\rho, x) = \rho^2 \widetilde{u}(\rho, x), \quad 0 < x < b,$$

satisfying the initial conditions $\widetilde{u}(\rho, 0) = u_0, \ \widetilde{u}'(\rho, 0) = u_1$.

Example

Denote by $e^0_{\mathfrak{J}_N}(\rho, x)$ the unique solution of

$$-y'' + \left(\sum_{k=1}^{N} \alpha_k \delta(x - x_k)\right) y = \rho^2 y, \quad 0 < x < b,$$

satisfying $e^0_{\mathfrak{J}_N}(\rho, 0) = 1$, $(e^0_{\mathfrak{J}_N})'(\rho, 0) = i\rho$. In this case we have $\widehat{s}_k(\rho, x) = \frac{\sin(\rho x)}{\rho}$ for $k = 1, \dots, N$. Hence, the solution $e^0_{\mathfrak{J}_N}(\rho, x)$ has the form

$$e^0_{\mathfrak{J}_N}(\rho, x) = e^{i\rho x} + \sum_{k=1}^{N} \alpha_k e^{i\rho x_k} H(x - x_k) \frac{\sin(\rho(x - x_k))}{\rho}$$

$$+ \sum_{J \in \mathcal{J}_N} \alpha_J H(x - x_{j_{|J|}}) e^{i\rho x_{j_1}} \left(\prod_{l=1}^{|J|-1} \frac{\sin(\rho(x_{j_{l+1}} - x_{j_l}))}{\rho}\right) \frac{\sin(\rho(x - x_{j_{|J|}}))}{\rho}.$$

Transmutation operators

- Let $h \in \mathbb{C}$. Denote by $\widetilde{e}_h(\rho, x)$ the unique solution of the regular equation satisfying $\widetilde{e}_h(\rho, 0) = 1$, $\widetilde{e}'_h(\rho, 0) = i\rho + h$.

[2] V. A. Marchenko, *Sturm-Liouville operators and applications*, Birkhäuser, Basel, 1986.

Transmutation operators

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- There exists a kernel[2] $\widetilde{K}^h \in C(\overline{\Omega}) \cap H^1(\Omega)$, where $\Omega = \{(x,t) \in \mathbb{R}^2 \,|\, 0 < x < b, |t| < x\}$, such that $\widetilde{K}^h(x,x) = \frac{h}{2} + \frac{1}{2}\int_0^x q(s)ds$, $\widetilde{K}^h(x,-x) = \frac{h}{2}$ and

$$\widetilde{e}_h(\rho, x) = e^{i\rho x} + \int_{-x}^{x} \widetilde{K}^h(x,t) e^{i\rho t} dt$$

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- For each $k \in \{1, \ldots, N\}$ there exists a kernel $\widehat{H}_k \in C(\overline{\Omega_k}) \cap H^1(\Omega_k)$ with $\Omega_k = \{(x,t) \in \mathbb{R}^2 \,|\, 0 < x < b - x_k,\ |t| \leqslant x\}$, and $\widehat{H}_k(x,x) = \frac{1}{2}\int_{x_k}^{x+x_k} q(s)ds,\ \widehat{H}_k(x,-x) = 0$, such that

$$\widehat{s}_k(\rho, x) = \frac{\sin(\rho x)}{\rho} + \int_0^x \widehat{H}_k(x,t)\frac{\sin(\rho t)}{\rho}dt$$

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$$\widehat{s}_k(\rho, x) = \frac{\sin(\rho x)}{\rho} + \int_0^x \widehat{H}_k(x,t)\frac{\sin(\rho t)}{\rho}dt$$

- From this we obtain the representation

$$\widehat{s}_k(\rho, x - x_k) = \int_{-(x-x_k)}^{x-x_k} \widetilde{K}_k(x,t)e^{i\rho t}dt,$$

where

$$\widetilde{K}_k(x,t) = \frac{1}{2}\chi_{x-x_k}(t) + \frac{1}{2}\int_{|t|}^{x-x_k} \widehat{H}_k(x - x_k, s)ds.$$

- The unique solution $e^h_{\mathfrak{J}_N}(\rho, x)$ of the eq. with point interactions which satisfies the initial conditions $e^h_{\mathfrak{J}_N}(\rho, 0) = 1$, $(e^h_{\mathfrak{J}_N})'(\rho, 0) = i\rho + h$ is given by

$$e^h_{\mathfrak{J}_N}(\rho, x) = \widetilde{e}_h(\rho, x) + \sum_{k=1}^{N} \alpha_k \widetilde{e}_h(\rho, x_k) H(x - x_k) \widehat{s}_k(\rho, x - x_k)$$

$$+ \sum_{J \in \mathcal{J}_N} \alpha_J H(x - x_{j_{|J|}}) \widetilde{e}_h(\rho, x_{j_1}) \left(\prod_{l=1}^{|J|-1} \widehat{s}_{j_l}(\rho, x_{j_{l+1}} - x_{j_l}) \right) \widehat{s}_{j_{|J|}}(\rho, x - x_j$$

Theorem

There exists a kernel $K^h_{\mathfrak{J}_N}(x,t)$ *defined on* Ω *such that*

$$e^h_{\mathfrak{J}_N}(\rho,x) = e^{i\rho x} + \int_{-x}^{x} K^h_{\mathfrak{J}_N}(x,t)e^{i\rho t}dt. \tag{6}$$

For any $0 < x \leqslant b$, $K^h_{\mathfrak{J}_N}(x,t)$ *is piecewise absolutely continuous with respect to the variable* $t \in [-x,x]$ *and satisfies* $K^h_{\mathfrak{J}_N}(x,\cdot) \in L_2(-x,x)$. *Furthermore,* $K^h_{\mathfrak{J}_N} \in L_\infty(\Omega)$.

The explicit form of the kernel is

$$\begin{aligned}
K^h_{\mathfrak{I}_N}(x,t) &= \chi_x(t)\widetilde{K}^h(x,t) \\
&+ \sum_{k=1}^{n} \alpha_k H(x-x_k)\Big(\chi_{[2x_k-x,x]}(t)\widetilde{K}_k(x,t-x_k) + \chi_{x_k}(t)\widetilde{K}^h(x_k,t) * \chi_{x-x_k}(t)\widetilde{K}_k(x,t)\Big) \\
&+ \sum_{J\in\mathcal{J}_N} \alpha_J H(x-x_{j_{|J|}})\left(\prod_{l=1}^{|J|-1}\right)^{*}\Big(\chi_{x_{j_{l+1}}-x_{j_l}}(t)\widetilde{K}_{j_l}(x_{j_{l+1}},t)\Big) \\
&\quad * \Big(\chi_{x-(x_{j_{|J|}}-x_{j_1})}(t)\widetilde{K}_{j_{|J|}}(x,t-x_{j_1}) + \chi_{x_{j_1}}(t)\widetilde{K}^h(x_{j_1},t) * \chi_{x-x_{j_{|J|}}}(t)\widetilde{K}_{j_{|J|}}(x,t)\Big).
\end{aligned}$$

Example

Consider the equation $-y'' + \alpha_1\delta(x - x_1)y = \rho^2 y$. In this case the solution $e^0_{\mathfrak{J}_1}(\rho, x)$ is given by

$$e^0_{\mathfrak{J}_1}(\rho, x) = e^{i\rho x} + \alpha_1 e^{i\rho x_1} H(x - x_1)\frac{\sin(\rho(x - x_1))}{\rho}.$$

We have

$$e^{i\rho x_1}\frac{\sin(\rho(x - x_1))}{\rho} = \frac{1}{2}\int_{x_1 - x}^{x - x_1} e^{i\rho(t + x_1)}dt = \frac{1}{2}\int_{2x_1 - x}^{x} e^{i\rho t}dt.$$

Hence $e^0_{\mathfrak{J}_1}(\rho, x) = e^{i\rho x} + \int_{-x}^{x} K^0_{\mathfrak{J}_1}(x, t)e^{i\rho t}dt$ with

$$K^0_{\mathfrak{J}_1}(x, t) = \frac{\alpha_1}{2}H(x - x_1)\chi_{[2x_1 - x, x]}(t).$$

Example

Now we consider the equation with two interactions $\mathfrak{I}_2 = \{(\alpha_1, x_1), (\alpha_2, x_2)\}$. In this case, the solution $e^0_{\mathfrak{I}_2}(\rho, x)$ has the form

$$e^0_{\mathfrak{I}_2}(\rho, x) = e^{i\rho x} + \alpha_1 e^{i\rho x_1} H(x - x_1)\frac{\sin(\rho(x - x_1))}{\rho} + \alpha_2 e^{i\rho x_2} H(x - x_2)\frac{\sin(\rho(x -}{\rho}$$

$$+ \alpha_1\alpha_2 e^{i\rho x_1} H(x - x_2)\frac{\sin(\rho(x_2 - x_1))}{\rho}\frac{\sin(\rho(x - x_2))}{\rho},$$

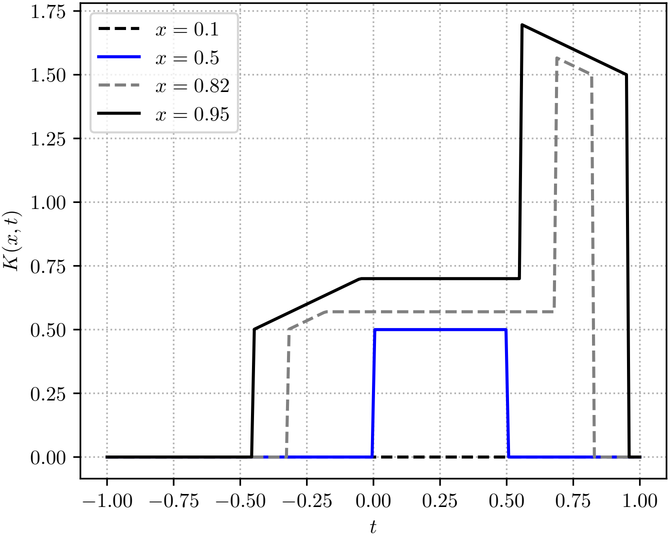
and the transmutation kernel $K^0_{\mathfrak{I}_2}(x, t)$ has the form

$$K^0_{\mathfrak{I}_2}(x, t) = \frac{\alpha_1 H(x - x_1)}{2}\chi_{[2x_1 - x, x]}(t) + \frac{\alpha_2 H(x - x_2)}{2}\chi_{[2x_1 - x, x]}(t)$$
$$+ \frac{\alpha_1\alpha_2 H(x - x_2)}{4}\left(\chi_{x_2 - x_1} * \chi_{x - x_2}\right)(t - x_1).$$

Direct computation shows that

$$\chi_{x_2-x_1} * \chi_{x-x_2}(t-x_1) =$$
$$\begin{cases} 0, & t \notin [2x_1 - x, x], \\ t + x - 2x_1, & 2x_1 - x < t < -|2x_2 - x - x_1| + x_1, \\ x - x_1 - |2x_2 - x - x_1|, & -|2x_2 - x - x_1| + x_1 < t < |2x_2 - x - x_1| + x_1 \\ x - t, & |2x_2 - x - x_1| + x_1 < t < x. \end{cases}$$

In the next figure, we can see some level curves of the kernel $K^0_{\mathfrak{J}_2}(x,t)$ (as a function of t), $\mathfrak{J}_2 = \{(0{,}25, 1), (0{,}75, 2)\}$, for some values of x.



$x = 0.1$
$x = 0.5$
$x = 0.82$
$x = 0.95$
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
$K(x,t)$
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
t

Proposition

The integral transmutation kernel $K^h_{\mathfrak{J}_N}$ satisfies the following Goursat conditions for $x \in [0,b]$

$$K^h_{\mathfrak{J}_N}(x,x) = \frac{1}{2}\left(h + \int_0^x q(s)ds + \sigma_{\mathfrak{J}_N}(x)\right) \quad \text{and} \quad K^h_{\mathfrak{J}_N}(x,-x) = \frac{h}{2}, \tag{7}$$

where

$$\sigma_{\mathfrak{J}_N}(x) := \sum_{k=1}^{N} \alpha_k H(x - x_k).$$

Thus, $2K^h_{\mathfrak{J}_N}(x,x)$ is a (distributional) antiderivative of the potential $q(x) + q_{\delta,\mathfrak{J}_N}(x)$.

Let $c^h_{\mathfrak{I}_N}(\rho,x)$ and $s_{\mathfrak{I}_N}(\rho,x)$ be the solutions of Eq. (1) satisfying the initial conditions

$$c^h_{\mathfrak{I}_N}(\rho,0)=1, \quad (c^h_{\mathfrak{I}_N})'(\rho,0)=h,$$
$$s_{\mathfrak{I}_N}(\rho,0)=0, \quad s'_{\mathfrak{I}_N}(\rho,0)=1.$$

Note that $c^h_{\mathfrak{I}_N}(\rho,x)=\frac{e^h_{\mathfrak{I}_N}(\rho,x)+e^h_{\mathfrak{I}_N}(-\rho,x)}{2}$ and $s_{\mathfrak{I}_N}(\rho,x)=\frac{e^h_{\mathfrak{I}_N}(\rho,x)-e^h_{\mathfrak{I}_N}(-\rho,x)}{2i\rho}$. Hence

$$c^h_{\mathfrak{I}_N}(\rho,x)=\cos(\rho x)+\int_0^x G^h_{\mathfrak{I}_N}(x,t)\cos(\rho t)dt,$$
$$s_{\mathfrak{I}_N}(\rho,x)=\frac{\sin(\rho x)}{\rho}+\int_0^x S_{\mathfrak{I}_N}(x,t)\frac{\sin(\rho t)}{\rho}dt,$$

where

$$G^h_{\mathfrak{I}_N}(x,t)=K^h_{\mathfrak{I}_N}(x,t)+K^h_{\mathfrak{I}_N}(x,-t),$$
$$S_{\mathfrak{I}_N}(x,t)=K^h_{\mathfrak{I}_N}(x,t)-K^h_{\mathfrak{I}_N}(x,-t).$$

The SPPS Method

- Let $f \in \mathcal{D}_2(\mathbf{L}_{q,\mathfrak{I}_N})$ be a nonvanishing solution of equation $\mathbf{L}_{q,\mathfrak{I}_N} f = 0$.

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- Let $f \in \mathcal{D}_2(\mathbf{L}_{q,\mathfrak{I}_N})$ be a nonvanishing solution of equation $\mathbf{L}_{q,\mathfrak{I}_N} f = 0$.
- We define the following recursive integrals: $\widetilde{X}^{(0)} \equiv X^{(0)} \equiv 1$, and for $k \in \mathbb{N}$

$$\widetilde{X}^{(k)}(x) := k \int_0^x \widetilde{X}^{(k-1)}(s) \left(f^2(s)\right)^{(-1)^{k-1}} ds, \tag{8}$$

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$$X^{(k)}(x) := k \int_0^x X^{(k-1)}(s) \left(f^2(s)\right)^{(-1)^{k}} ds. \tag{9}$$

- The functions $\{\varphi_f^{(k)}(x)\}_{k=0}^{\infty}$ defined by

$$\varphi_f^{(k)}(x) := \begin{cases} f(x)\widetilde{X}^{(k)}(x), & \text{if } k \text{ even}, \\ f(x)X^{(k)}(x), & \text{if } k \text{ odd}. \end{cases} \tag{10}$$

for $k \in \mathbb{N}_0$, are called the *formal powers* associated to f.

Theorem (SPPS method)

The functions

$$u_0(\rho,x)=\sum_{k=0}^{\infty}\frac{(-1)^k\rho^{2k}\varphi_f^{(2k)}(x)}{(2k)!},\quad u_1(\rho,x)=\sum_{k=0}^{\infty}\frac{(-1)^k\rho^{2k}\varphi_f^{(2k+1)}(x)}{(2k+1)!}$$

belong to $\mathcal{D}_2\left(\mathbf{L}_{q,\mathfrak{I}_N}\right)$, *and* $\{u_0(\rho,x),u_1(\rho,x)\}$ *is a fundamental set of solutions for the equation with point interactions, satisfying the initial conditions*

$$u_0(\rho,0)=f(0), u_0'(\rho,0)=f'(0), \tag{11}$$

$$u_1(\rho,0)=0, u_1'(\rho,0)=\frac{1}{f(0)}, \tag{12}$$

The series converge absolutely and uniformly on $x\in[0,b]$, *the series of the derivatives converge in* $L_2(0,b)$ *and the series of the second derivatives converge in* $L_2(x_j,x_{j+1})$, $j=0,\cdots,N$.

With respect to ρ the series converge absolutely and uniformly on any compact subset of the complex ρ-plane.

- The proof of the convergence is given by the estimates of the form

$$|\widetilde{X}^{(n)}(x)| \leqslant M_1^n b^n,\ |X^{(n)}(x)| \leqslant M_1^n b^n \quad \text{for all } x \in [0,b],$$

  and the relations for the derivatives:

$$D\varphi_f^{(k)} = \frac{f'}{f}\varphi_f^{(k)} + k\varphi_{\frac{1}{f}}^{(k-1)}$$
$$D^2\varphi_f^{(k)} = \frac{f''}{f}\varphi_f^{(k)} + k(k-1)\varphi_f^{(k-2)}$$

- The formal powers satisfy the conditions

  $\mathbf{L}_{q,\mathfrak{I}_N}\varphi_f^{(k)} = 0,\ k = 0,1,$ and $\mathbf{L}_{q,\mathfrak{I}_N}\varphi_f^{(k)} = -k(k-1)\varphi_f^{(k-2)},\ k \geq 2,$

  that is, $\{\varphi_f^{(k)}\}_{k=0}^{\infty}$ is an $-\mathbf{L}_{q,\mathfrak{I}_N}$-base.

Proposition

Let $\{u, v\} \in \mathcal{D}_2\left(\mathbf{L}_{q,\mathfrak{I}_N}\right)$ be a fundamental set of solutions for (1). Then there exist constants $c_1, c_2 \in \mathbb{C}$ such that the solution $f = c_1 u + c_2 v$ does not vanish in the whole segment $[0, b]$.

Consequently, there exists a pair of constants $(c_1, c_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$ such that

$$y_0(x) = c_1 + c_2 x + \sum_{k=1}^{N} \alpha_k (c_1 + c_2 x_k) H(x - x_k)(x - x_k)$$
$$+ \sum_{J \in \mathcal{J}_N} \alpha_J (c_1 + c_2 x_{j_1}) H(x - x_{j_{|J|}}) \left(\prod_{l=1}^{|J|-1} (x_{j_{l+1}} - x_{j_1}) \right) (x - x_{j_{|J|}})$$

is a non-vanishing solution of the the equation with purely distributional potential for $\rho = 0$ (if $\alpha_1, \ldots, \alpha_k \in (0, \infty)$, it is enough with choosing $c_1 = 1$, $c_2 = 0$).

Theorem

Define the recursive integrals $\{Y^{(k)}\}_{k=0}^{\infty}$ *and* $\{\tilde{Y}^{(k)}\}_{k=0}^{\infty}$ *as follows:* $Y^{(0)} \equiv \tilde{Y}^{(0)} \equiv 1$, *and for* $k \geqslant 1$

$$Y^{(k)}(x) = \begin{cases} \int_0^x Y^{(k)}(s)q(s)y_0^2(s)ds, & \text{if } k \text{ is even}, \\ \int_0^x \frac{Y^{(k)}(s)}{y_0^2(s)}ds, & \text{if } k \text{ is odd}, \end{cases} \tag{13}$$

$$\tilde{Y}^{(k)}(x) = \begin{cases} \int_0^x \tilde{Y}^{(k)}(s)q(s)y_0^2(s)ds, & \text{if } k \text{ is odd}, \\ \int_0^x \frac{\tilde{Y}^{(k)}(s)}{y_0^2(s)}ds, & \text{if } k \text{ is even}. \end{cases} \tag{14}$$

Define

$$f_0(x) = y_0(x)\sum_{k=0}^{\infty}\tilde{Y}^{(2k)}(x), \qquad f_1(x) = y_0(x)\sum_{k=0}^{\infty}Y^{(2k+1)}(x). \tag{15}$$

Then $\{f_0, f_1\} \subset \mathcal{D}_2(\mathbf{L}_{q,\mathfrak{I}_N})$ is a fundamental set of solution for $\mathbf{L}_{q,\mathfrak{I}_N} u = 0$ satisfying the initial conditions $f_0(0) = c_1$, $f_0'(0) = c_2$, $f_1(0) = 0$, $f_1'(0) = 1$. Both series converge uniformly and absolutely on $x \in [0, b]$. The series of the derivatives converge in $L_2(0, b)$, and on each interval $[x_j, x_{j+1}]$, $j = 0, \ldots, N$, the series of the second derivatives converge in $L_2(x_j, x_{j+1})$. Hence there exist constants $C_1, C_2 \in \mathbb{C}$ such that $f = C_1 f_0 + C_2 f_1$ is a non-vanishing solution of $\mathbf{L}_{q,\mathfrak{I}_N} u = 0$ in $[0, b]$.

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- Suppose that $f(0) = 1$ and set $h = f'(0)$.

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- Let $K^f_{\mathfrak{I}_N}$ be the transmutation kernel associated to $h = f'(0)$ and define the operator

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- Note that $\mathbf{T}_{\mathfrak{I}_N}^f \in \mathcal{B}(L_2(-b,b), L_2(0,b))$.

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- $e^f_{\mathfrak{I}_N}(\rho, x) = \mathbf{T}^f_{\mathfrak{I}_N}[e^{i\rho x}]$.
- By the SPPS method,

$$e^h_{\mathfrak{I}_N}(\rho, x) = \sum_{k=0}^{\infty} \frac{(i\rho)^k \varphi_f^{(k)}(x)}{k!}.$$

- Substituting the Taylor series of the exponential in $\mathbf{T}^{f}_{\mathfrak{I}_N}[e^{i\rho x}]$ and comparing with the SPPS series we obtain:

- Substituting the Taylor series of the exponential in $\mathbf{T}_{\mathfrak{I}_N}^{f}[e^{i\rho x}]$ and comparing with the SPPS series we obtain:

Theorem

The transmutation operator $\mathbf{T}_{\mathfrak{I}_N}^{f}$ *satisfies the following relations*

$$\mathbf{T}_{\mathfrak{I}_N}^{f}\left[x^k\right] = \varphi_f^{(k)}(x) \qquad \forall k \in \mathbb{N}_0. \tag{16}$$

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The transmutation operator $\mathbf{T}^f_{\mathfrak{I}_N}$ *satisfies the following relations*

$$\mathbf{T}^f_{\mathfrak{I}_N}\left[x^k\right] = \varphi_f^{(k)}(x) \qquad \forall k \in \mathbb{N}_0. \tag{16}$$

- Since $\{\varphi_f^{(k)}\}_{k=0}^{\infty}$ are an $-\mathbf{L}_{q,\mathfrak{I}_N}$-base, by linearity we get the transmutation relation

$$\mathbf{L}_{q,\mathfrak{I}_N}\mathbf{T}^f_{\mathfrak{I}_N} p = -\mathbf{T}_{\mathfrak{I}_N} D^2 p$$

for all $p \in \mathcal{P}[-b,b] = \mathrm{Span}\{x^k\}_{k=0}^{\infty}$.

- Relation $\mathbf{L}_{q,\mathfrak{J}_N}\mathbf{T}^f_{\mathfrak{J}_N}p = -\mathbf{T}_{\mathfrak{J}_N}D^2$ can be written as

$$\mathbf{T}^f_{\mathfrak{J}_N}p(x) = p(0)\varphi_f^{(0)} + p'(0)\varphi_f^{(1)}(x) - f(x)\int_0^x \frac{1}{f^2(t)}\int_0^t f(s)\mathbf{T}^f_{\mathfrak{J}_N}p''(s)dsdt$$

- Relation $\mathbf{L}_{q,\mathfrak{I}_N}\mathbf{T}^f_{\mathfrak{I}_N}p = -\mathbf{T}_{\mathfrak{I}_N}D^2$ can be written as

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- Since the operators involved are bounded and $\mathcal{P}[-b,b]$ is dense in $H^2(-b,b)$ we get the following result

- Relation $\mathbf{L}_{q,\mathfrak{I}_N}\mathbf{T}^f_{\mathfrak{I}_N}p = -\mathbf{T}_{\mathfrak{I}_N}D^2$ can be written as

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- Since the operators involved are bounded and $\mathcal{P}[-b,b]$ is dense in $H^2(-b,b)$ we get the following result

Theorem

The operator $\mathbf{T}^f_{\mathfrak{I}_N}$ is a transmutation operator for the pair $\mathbf{L}_{q,\mathfrak{I}_N}$, $-D^2$ in $H^2(-b,b)$, that is, $\mathbf{T}^f_{\mathfrak{I}_N}\left(H^2(-b,b)\right) \subset \mathcal{D}_2\left(\mathbf{L}_{q,\mathfrak{I}_N}\right)$ and

$$\mathbf{L}_{q,\mathfrak{I}_N}\mathbf{T}_{\mathfrak{I}_N}u = -\mathbf{T}_{\mathfrak{I}_N}D^2u \qquad \forall u \in H^2(-b,b)$$

Fourier-Legendre and NSBF expansions

- For $x \in (0,b]$ fixed, $K^{f}_{\mathfrak{I}_N}(x,\cdot) \in L_2(-x,x)$, hence it admits a Fourier series in the orthonormal basis $\left\{P_n\left(\frac{t}{x}\right)\right\}_{n=0}^{\infty}$, where $\{P_n(\tau)\}_{n=0}^{\infty}$ are the Legendre polynomials.

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- Hence $K^h_{\mathfrak{J}_N}(x,t) = \displaystyle\sum_{n=0}^{\infty} \frac{a_n(x)}{x} P_n\left(\frac{t}{x}\right)$.
- The coefficients are given by

$$a_n(x) = \left(n + \frac{1}{2}\right) \int_{-x}^{x} K^h_{\mathfrak{J}_N}(x,t) P_n\left(\frac{t}{x}\right) dt \qquad \forall n \in \mathbb{N}_0$$

Example

Consider the kernel $K^0_{\mathfrak{J}_1}(x,t) = \frac{\alpha_1}{2} H(x-x_1)\chi_{[2x_1-x,x]}$. In this case, the Fourier-Legendre coefficients have the form

$$\begin{aligned} a_n(x) &= \frac{\alpha_1}{2}\left(n+\frac{1}{2}\right) H(x-x_1)\int_{2x_1-x}^{x} P_n\left(\frac{t}{x}\right) dt \\ &= \frac{\alpha_1}{2}\left(n+\frac{1}{2}\right) xH(x-x_1)\int_{2\frac{x_1}{x}-1}^{1} P_n(t)dt. \end{aligned}$$

From this we obtain $a_0(x) = \frac{\alpha_1}{2} H(x-x_1)(x-x_1)$. Using formula $P_n(t) = \frac{1}{2n+1}\frac{d}{dt}\left(P_{n+1}(t) - P_{n-1}(t)\right)$ for $n \in \mathbb{N}$, and that $P_n(1) = 0$ for all $n \in \mathbb{N}$, we have

$$a_n(x) = \frac{\alpha_1}{4} xH(x-x_1)\left[P_{n-1}\left(\frac{2x_1}{x}-1\right) - P_{n+1}\left(\frac{2x_1}{x}-1\right)\right].$$

- Similar representations can be obtained for the *cosine and sine* kernels:

$$G^h_{\mathfrak{I}_N}(x,t) = \sum_{n=0}^{\infty} \frac{g_n(x)}{x} P_{2n}\left(\frac{t}{x}\right),$$

$$S_{\mathfrak{I}_N}(x,t) = \sum_{n=0}^{\infty} \frac{s_n(x)}{x} P_{2n+1}\left(\frac{t}{x}\right),$$

where the coefficients are given by

$$g_n(x) = 2a_{2n}(x) = (4n+1)\int_0^x G^h_{\mathfrak{I}_N}(x,t) P_{2n}\left(\frac{t}{x}\right) dt,$$

$$s_n(x) = 2a_{2n+1} = (4n+3)\int_0^x S_{\mathfrak{I}_N}(x,t) P_{2n+1}\left(\frac{t}{x}\right) dt.$$

- For every $n \in \mathbb{N}_0$ we write the Legendre polynomial $P_n(z)$ in the form $P_n(z) = \sum_{k=0}^{n} l_{k,n} z^k$.
- Note that if n is even, $l_{k,n} = 0$ for odd k, and $P_{2n}(z) = \sum_{k=0}^{n} \tilde{l}_{k,n} z^{2k}$ with $\tilde{l}_{k,n} = l_{2k,2n}$.
- Similarly $P_{2n+1}(z) = \sum_{k=0}^{n} \hat{l}_{k,n} z^{2k+1}$ with $\hat{l}_{k,n} = l_{2k+1,2n+1}$.

Proposition

The coefficients $\{a_n(x)\}_{n=0}^{\infty}$ of the Fourier-Legendre expansion of the canonical transmutation kernel $K_{\mathfrak{I}_N}^{f}(x,t)$ are given by

$$a_n(x) = \left(n + \frac{1}{2}\right)\left(\sum_{k=0}^{n} l_{k,n} \frac{\varphi_f^{(k)}(x)}{x^k} - 1\right).$$

The coefficients of the canonical cosine and sine kernels satisfy the following relations for all $n \in \mathbb{N}_0$

$$g_n(x) = (4n+1)\left(\sum_{k=0}^{n} \tilde{l}_{k,n} \frac{\varphi_f^{(2k)}(x)}{x^{2k}} - 1\right),$$

$$s_n(x) = (4n+3)\left(\sum_{k=0}^{n} \hat{l}_{k,n} \frac{\varphi_f^{(2k+1)}(x)}{x^{2k+1}} - 1\right).$$

Theorem

The solutions $c^h_{\mathfrak{J}_N}(\rho, x)$ and $s_{\mathfrak{J}_N}(\rho, x)$ admit the following NSBF representations

$$c^h_{\mathfrak{J}_N}(\rho, x) = \cos(\rho x) + \sum_{n=0}^{\infty}(-1)^n g_n(x) j_{2n}(\rho x),$$

$$s_{\mathfrak{J}_N}(\rho, x) = \frac{\sin(\rho x)}{\rho} + \frac{1}{\rho}\sum_{n=0}^{\infty}(-1)^n s_n(x) j_{2n+1}(\rho x),$$

where j_ν stands for the spherical Bessel function $j_\nu(z) = \sqrt{\frac{\pi}{2z}} J_{\nu+\frac{1}{2}}(z)$. The series converge pointwise with respect to x in $(0, b]$ and uniformly with respect to ρ on any compact subset of the complex ρ-plane.

Moreover, for $M \in \mathbb{N}$ the functions

$$c^h_{\mathfrak{I}_N,M}(\rho,x) = \cos(\rho x) + \sum_{n=0}^{M} (-1)^n g_n(x) j_{2n}(\rho x),$$

$$s_{\mathfrak{I}_N,M}(\rho,x) = \frac{\sin(\rho x)}{\rho} + \frac{1}{\rho}\sum_{n=0}^{M} (-1)^n s_n(x) j_{2n+1}(\rho x),$$

obey the estimates

$$|c^h_{\mathfrak{I}_N}(\rho,x) - c^h_{\mathfrak{I}_N,M}(\rho,x)| \leqslant 2\epsilon_{2M}(x)\sqrt{\frac{\sinh(2bC)}{C}},$$

$$|\rho s_{\mathfrak{I}_N}(\rho,x) - \rho s_{\mathfrak{I}_N,M}(\rho,x)| \leqslant 2\epsilon_{2M+1}(x)\sqrt{\frac{\sinh(2bC)}{C}},$$

for any $\rho \in \mathbb{C}$ belonging to the strip $|\operatorname{Im}\rho| \leqslant C$, $C > 0$, and where $\epsilon_M(x) = \|K^h_{\mathfrak{I}_N}(x,\cdot) - K^h_{\mathfrak{I}_N,2M}(x,\cdot)\|_{L_2(-x,x)}$.

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$$\psi_{\mathfrak{I}_N}^H(\rho, b) = 1, \quad (\psi_{\mathfrak{I}_N}^H)'(\rho, b) = -H,$$
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- The following relations hold

$$g_0(x) = c^h_{\mathfrak{I}_N}(0, x) - 1, \quad s_0(x) = 3\left(\frac{s_{\mathfrak{I}_N}(0, x)}{x} - 1\right).$$

What's next?

- **Solution of direct spectral problems**. For example, the solution of the Sturm-Liouville problem with the Dirichlet-to-Dirichlet conditions is reduced to compute the zeros of the characteristic function

$$0 = \rho s_{\mathfrak{I}_N}(\rho, b) = \sin(\rho b) + \sum_{n=0}^{\infty} (-1)^n s_n(b) j_{2n+1}(\rho b)$$

[3] Similar to the procedure used in the regular case V. V. Kravchenko, L.J. Navarro, S.M. Torba, *Representation of solutions to the one-dimensional Schrödinger equation in terms of Neumann series of Bessel functions.* Appl. Math. Comput. 314(1) (2017) 173-192.

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- The coefficientes $\{\alpha_n(x)\}_{n=0}^{\infty}$ can be computed by a recursive integration procedure [3]

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- **Practical solution of inverse problems**
  - Gelfand-Levitan equation (eigenvalues+normalizing constants)
    Substitution of the Fourier-Legendre series of the integral kernel reduces the problem to solve a linear system of algebraic equations where the unknowns are the coefficientes $\{\alpha_n\}$.

[4] We follow the ideas presented in V. V. Kravchenko, *Spectrum completion and inverse Sturm–Liouville problems*. Mathematical Methods in the Applied Sciences, 2023, v. 46, issue 5, 5821–5835. doi:10.1002/mma.8869

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  - Problems with 2 spectrums For example, let $\{\mu_k^2\}_{k=1}^{\infty}$ and $\{\rho_k^2\}_{k=1}^{\infty}$ be the spectums of problems with D-D and D-N conditions. The problem can be reduced to solve a system of the form

$$s_{\mathfrak{I}_N}(\rho_k, x) = \beta_k \psi^0_{\mathfrak{I}_N}(\rho_k, x),$$

where β_k are known and the unknowns are the Fourier-Legendre coefficientes of the NSBF series of $s_{\mathfrak{I}_N}$ and $\psi^0_{\mathfrak{I}_N}$ [4].

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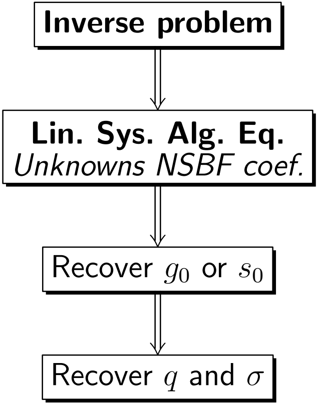
- The potential q can be recovered from g_0 and s_0 on each interval (x_k, x_{k+1}).
- The function $\sigma = \int_0^x q(s)ds + \sigma_{\mathfrak{I}_N}$ can be recovered from $f = g_0 + 1$ by the relation

$$\sigma(x) = \frac{f'(x)}{f(x)} + \int_{x_0}^{x} \left(\frac{f'(t)}{f(t)} \right)^2 dt.$$


Inverse problem
Lin. Sys. Alg. Eq.
Unknowns NSBF coef.
Recover g_0 or s_0
Recover q and σ

The results of this talk are found in

V. V. Kravchenko, V. A. Vicente-Benítez, *Schrödinger equation with finitely many δ-interactions: closed form, integral and series representations for the solutions*, Anal. Math. Phys. 14, 97 (2024) https://doi.org/10.1007/s13324-024-00957-4

Xquixhe pe' laatu!

(Gracias por su atención)

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