Multi-term Fractional Integro-Differential Equations in Spaces of Average Power Growth Functions

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Functions of Bounded Square Average

On $\mathbb{R}$

N. Wiener. Generalized harmonic analysis. Acta Math. 55 (1930), 117-258.

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 dt < \infty$$

$L_2(R)$, periodic functions, almost periodic functions

On R_+

$$f \in BSA_1(\mathbb{R}_+) \iff \sup_{T>0} \frac{1}{T+1} \int_0^T |f(t)|^2 dt < \infty$$

Functions with Bounded Square Average

How to characterize such functions

Paley-Wiener Theorem

Laplace Transform and its inverse

$$\begin{aligned} F(s) &= (\mathcal{L}f)(s) := \int_0^\infty e^{-st} f(t)\, dt, \\ f(t) &= (\mathcal{L}^{-1}F)(t) := \frac{1}{2\pi i} \int_{\mathrm{Re}\ s=d} e^{st} F(s)\, ds \end{aligned} \tag{1}$$

Paley-Wiener Theorem: $f \in L_2(\mathbb{R}_+) \iff F(s) = F(x+iy)$ is holomorphic in the right half plane $y > 0$ and

$$\sup_{y>0} \int_{-\infty}^{\infty} |F(x+iy)|^2 dx < \infty$$

Characterization of $BSA_1(\mathbb{R}_+)$

Theorem

A function $F(s)$ is the Laplace transform of $f \in BSA_1(\mathbb{R}_+)$ if and only if $F(s)$ is analytic in the right-half plane $\mathrm{Re}\ s > 0$, and

$$\sup_{x>0} \frac{x}{x+1} \int_{-\infty}^{\infty} |F(x+iy)|^2 \, dy < \infty. \tag{2}$$

Corollary

Let $F(s)$ be analytic in the right half plane $\operatorname{Re} s > 0$ and $|F(s)| \le C|s|^{-\alpha}$, $\frac{1}{2} < \alpha \le 1$. Then F is the Laplace transform of a function $f \in BSA_1(\mathbb{R}_+)$.

Theorem

Let $\|g\|_1 < k$, $0 < \alpha \le 1$, and $G(s) = (\mathcal{L}g)(s)$, then

$$\mathcal{L}^{-1}\left(\frac{s^{\alpha-1}}{s^\alpha + k + G(s)}\right) \in BSA_1(\mathbb{R}_+). \tag{3}$$

If moreover, $\frac{1}{2} < \alpha \le 1$, then

$$\mathcal{L}^{-1}\left(\frac{1}{s^\alpha + k + G(s)}\right) \in BSA_1(\mathbb{R}_+) \tag{4}$$

Convolution

Theorem

Let $f \in BSA_1(\mathbb{R}_+)$ and $g \in L_1(\mathbb{R}_+)$. Then their Laplace convolution

$$h(t) = (f * g)(t) := \int_0^t f(t-\tau)\, g(\tau)\, d\tau \tag{5}$$

belongs to $BSA_1(\mathbb{R}_+)$.

The Tauberian theorem for the Laplace transform

$$f(t) \sim \frac{At^{p-1}}{\Gamma(p)}, \quad t \to \infty \quad \Longrightarrow \quad F(s) \sim \frac{A}{s^p} \quad s \to 0_+ \quad p > 0, \quad (6)$$

says that, if $f(t)$ grows as t^{p-1} at infinity, then $F(s)$ grows as s^{-p} at 0.

The converse question is if $F(s)$ is holomorphic in the right-half plane $\operatorname{Re} s > 0$, and has a power growth at 0, whether it is the Laplace transform of a power growth function. The answer turns out affirmative if we consider functions of square average power growth instead of functions with pointwise power growth.

Definition

By $BSA_p(\mathbb{R}_+)$, $p \geq 0$, we denote the set of locally integrable functions f on $\mathbb{R}_+$ such that

$$\sup_{T>0} \frac{1}{(T+1)^p} \int_0^T |f(t)|^2 \, dt < \infty, \tag{7}$$

and

$$BSA_\infty(\mathbb{R}_+) = \bigcup_{p>0} BSA_p(\mathbb{R}_+). \tag{8}$$

$f \in BSA_p^m(\mathbb{R}_+)$ if $f, f', ..., f^{(m)} \in BSA_p(\mathbb{R}_+)$.

Subspaces

$BSA_0(\mathbb{R}_+) = L_2(\mathbb{R}_+)$,
$BSA_p(\mathbb{R}_+) \subset BSA_{p'}(\mathbb{R}_+)$ if $p < p'$,
$L_2(\mathbb{R}_+) \cup L_\infty(\mathbb{R}_+) \subset BSA_p(\mathbb{R}_+)$, for any $p \geq 1$.
$L_q(\mathbb{R}_+) \subset BSA_p(\mathbb{R}_+)$, for $2 \leq q \leq \infty, p \geq 1$.
However, note that, for $p \geq 0$, we have $f(t) = t^p \in BSA_{2p+1}(\mathbb{R}_+)$, and yet $f(t) \notin L_q(\mathbb{R}_+), 0 < q < \infty$.

Characterization of $BSA_p(\mathbb{R}_+)$

Theorem

A function $F(s)$ is the Laplace transform of $f \in BSA_p(\mathbb{R}_+)$ if, and only if, $F(s)$ is holomorphic in the right-half plane $\mathrm{Re}\, s > 0$, and

$$\sup_{x>0} \left(\frac{x}{x+1}\right)^p \int_{-\infty}^{\infty} |F(x+iy)|^2 \, dy < \infty. \tag{9}$$

The case $p = 0$ is the Paley-Wiener theorem for the Laplace transform

Corollary

Let $F(s)$ be holomorphic in the right half plane $\mathrm{Re}\, s > 0$ and $|F(s)| \le C|s|^{-\alpha}$, $\alpha > \frac{1}{2}$. Then F is the Laplace transform of a function $f \in BSA_{2\alpha-1}(\mathbb{R}_+)$.

Corollary

Let $\|g\|_1 < k$, if $0 < \alpha \le 1$, and $\|g\|_1 < \dfrac{k|\tan(\alpha\frac{\pi}{2})|}{1+|\tan(\alpha\frac{\pi}{2})|}$, if $1 < \alpha < 2$, and $\beta < \alpha - \frac{1}{2}$, then

$$\mathcal{L}^{-1}\left(\frac{s^\beta}{s^\alpha + k + G(s)}\right) \in BSA_{2(\alpha-\beta)-1}(\mathbb{R}_+). \tag{10}$$

Two-parametric Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \quad \alpha > 0.$$

We have

$$\mathcal{L}(t^{\beta-1} E_{\alpha,\beta}(-kt^{\alpha}))(s) = \frac{s^{\alpha-\beta}}{s^{\alpha} + k}.$$

Consequently, if $k > 0, 0 < \alpha \leq 1$, and $\beta > \frac{1}{2}$, then

$$t^{\beta-1} E_{\alpha,\beta}(-kt^{\alpha}) \in BSA_{2\beta-1}(\mathbb{R}_+).$$

Prabhakar Function

The Prabhakar (three-parametric Mittag-Leffler) function

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{(\gamma)_j}{j!\Gamma(\alpha j+\beta)} z^j, \quad \operatorname{Re}\alpha > 0,\ \operatorname{Re}\beta > 0,\ \gamma > 0,$$

where $(\gamma)_j = \gamma(\gamma+1)\cdots(\gamma+j-1)$.
The Laplace transform of the Prabhakar function has the form

$$\mathcal{L}(t^{\beta-1}E^{\gamma}_{\alpha,\beta}(\lambda t^{\alpha}))(s) = \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}-\lambda)^{\gamma}},$$

which is valid for all $\operatorname{Re}\alpha > 0,\ \operatorname{Re}\beta > 0,\ \gamma > 0,\ \lambda \in \mathbb{C}$, such that $|\lambda s^{-\alpha}| < 1$.
Consequently, if k, $\alpha > 0$, and $\beta > \frac{1}{2}$, then $t^{\beta-1}E^{\gamma}_{\alpha,\beta}(-kt^{\alpha}) \in BSA_{2\beta-1}(\mathbb{R}_+)$.

Multi-term Case

Theorem

Let $g \in L_1(\mathbb{R}_+)$, $0 < \alpha_n < ... < \alpha_1 < \alpha_0$, *and* $a_1, a_2, ..., a_n > 0$. *Let* $\|g\|_1 < k$, *if* $\frac{1}{2} < \alpha_0 \le 1$, *and* $\|g\|_1 < \dfrac{k|\tan(\alpha_0 \frac{\pi}{2})|}{1 + |\tan(\alpha_0 \frac{\pi}{2})|}$ *if* $1 < \alpha_0 < 2$. *Then the inverse Laplace transform*

$$\mathcal{L}^{-1}\left(\frac{1}{s^{\alpha_0} + \sum_{j=1}^{n} a_j s^{\alpha_j} + k + G(s)}\right) \in BSA_{2\alpha_0 - 1}(\mathbb{R}_+). \qquad (11)$$

Convolution

Lemma

Let $f \in BSA_p(\mathbb{R}_+)$ and $g \in L_1(\mathbb{R}_+)$. Then their Laplace convolution

$$h(t) = (f * g)(t) := \int_0^t f(t-\tau)\, g(\tau)\, d\tau \qquad (12)$$

belongs to $BSA_p(\mathbb{R}_+)$.

The Riemann-Liouville fractional derivative

$$D_{0+}^{\alpha} f(t) = \frac{d^n}{dt^n} I_{0+}^{n-\alpha} f(t), \tag{13}$$

$$I_{0+}^{n-\alpha} f(t) = \int_0^t \frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} f(\tau)\, d\tau, \quad \alpha < n. \tag{14}$$

$$\mathcal{L}\left(D_{0+}^{\alpha} f\right)(s) = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{n-k-1} D_{0+}^{\alpha+k-n} f(0+), \quad n-1 \leq \alpha < n. \tag{15}$$

Multi-Term Riemann-Liouville Fractional Integro-Differential Equation $\frac{1}{2} < \alpha_0 \leq 1$

$$D_{0+}^{\alpha_0} f(t) + \sum_{j=1}^{n} a_j D_{0+}^{\alpha_j} f(t) + kf(t) + \tag{16}$$

$$+ \int_0^t g(t-\tau) f(\tau) d\tau = h(t), \quad I_{0+}^{1-\alpha_0} f(0+) = f_0,$$

$$\frac{1}{2} < \alpha_0 \leq 1,\ 0 < \alpha_n < ... < \alpha_1 < \alpha_0,$$

where $k, a_1, a_2, ..., a_n \in \mathbb{R}_+,\ g, h \in L_1(\mathbb{R}_+)$ are given, and f is an unknown.

Theorem

Let $k>0$, $f_0\in\mathbb{R}$, $g,h\in L_1(\mathbb{R}_+)$, be given, and $\|g\|_1<k$. Then the multi-term Riemann-Liouville fractional integro-differential equation (16) has a unique solution f from $BSA_{2\alpha_0-1}(\mathbb{R}_+)$.

Denote

$$M(s)=\frac{1}{s^{\alpha_0}+\sum_{j=1}^{n}a_js^{\alpha_j}+k+G(s)}, \tag{17}$$

then

$$f(t)=f_0m(t)+\int_0^t m(t-\tau)h(\tau)\,d\tau. \tag{18}$$

Multi-Term Riemann-Liouville Fractional Integro-Differential Equation $\frac{3}{2} < \alpha_0 < 2$

Consider equation (16) when $\frac{3}{2} < \alpha_0 < 2$. The initial conditions

$$I_{0+}^{2-\alpha_0} f(0+) = f_0, \quad D_{0+}^{\alpha_0 - 1} f(0+) = f_1.$$

Theorem

Let $f_0, f_1 \in \mathbb{R}$, be given, and $\|g\|_1 < \dfrac{|\tan(\alpha_0 \frac{\pi}{2})| k}{1 + |\tan(\alpha_0 \frac{\pi}{2})|}$. Then the multi-term Riemann-Liouville fractional integro-differential equation (16) has a unique solution f from $BSA_{2\alpha_0 - 1}(\mathbb{R}_+)$.

Denote

$$M_1(s) = \frac{s}{s^{\alpha_0} + \sum_{j=1}^{n} a_j s^{\alpha_j} + k + G(s)} \tag{19}$$

$$\mu = \begin{cases} 1, & \text{if} \quad \alpha_0 - \alpha_1 = 1 \\ 0, & \text{if} \quad \alpha_0 - \alpha_1 > 1 \end{cases} .$$

then

$$f(t) = (f_1 + a_1 f_0 \mu)\, m(t) + f_0\, m_1(t) + \int_0^t m(t-\tau) h(\tau)\, d\tau. \tag{20}$$

The Caputo fractional derivative

$$
\begin{aligned}
{}^{\mathcal{C}}\partial_t^{\alpha} f(t) &= \int_0^t \frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} f^{(n)}(\tau)\, d\tau, \quad n-1<\alpha<n, \\
{}^{\mathcal{C}}\partial_t^{n} f(t) &= f^{(n)}(t).
\end{aligned} \tag{21}
$$

$$
\mathcal{L}\left({}^{\mathcal{C}}\partial_t^{\alpha} f\right)(s) = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \quad n-1<\alpha\le n. \tag{22}
$$

Multi-Term Caputo Fractional Integro-Differential Equation $\frac{1}{2} < \alpha_0 \le 1$

$$
{}^{\mathcal{C}}\partial_t^{\alpha_0} f(t) + \sum_{j=1}^{n} a_j {}^{\mathcal{C}}\partial_t^{\alpha_j} f(t) + kf(t) + \int_0^t g(t-\tau) f(\tau) d\tau = h(t),
$$

$$
f(0+) = f_0, \quad \frac{1}{2} < \alpha_0 \le 1,\ 0 < \alpha_n < \ldots < \alpha_1 < \alpha_0, \qquad (23)
$$

where $a_j, k \in \mathbb{R}_+, j = 1, 2, \ldots, n$, and $g, h \in L_1(\mathbb{R}_+)$ are given, and f is an unknown.

Explicit Solution in $BSA_{2(\alpha_0-\alpha_n)+1}(\mathbb{R}_+)$

Theorem

Let $k>0$, $f_0 \in \mathbb{R}$, $g, h \in L_1(\mathbb{R}_+)$, be given, and $\|g\|_1 < k$. Then the Caputo fractional integro-differential equation (23) has a unique solution f from $BSA_{2(\alpha_0-\alpha_n)+1}(\mathbb{R}_+)$.

Denote

$$L_j(s) = \frac{s^{\alpha_j - 1}}{s^{\alpha_0} + \sum_{i=1}^{n} a_i s^{\alpha_i} + k + G(s)}, \quad j = 0, 1, \cdots, n, \qquad (24)$$

then

$$f(t) = f_0\, l_0(t) + f_0 \sum_{j=1}^{n} a_j l_j(t) + \int_0^t m(t-\tau) h(\tau)\, d\tau. \qquad (25)$$

Multi-Term Caputo Fractional Integro-Differential Equation $1 < \alpha_0 \leq 2$

$$
\begin{aligned}
&{}^{\mathcal{C}}\partial_t^{\alpha_0} f(t) + \sum_{j=1}^{n} a_j {}^{\mathcal{C}}\partial_t^{\alpha_j} f(t) + kf(t) + \int_0^t g(t-\tau) f(\tau) d\tau = h(t), \\
&f(0+) = f_0,\ f'(0+) = f_1, \qquad (26) \\
&0 < \alpha_n < \cdots < \alpha_{l+1} < 1 < \alpha_l < \cdots < \alpha_1 < \alpha_0 \leq 2,
\end{aligned}
$$

where $a_j, k \in \mathbb{R}_+, j = 1, 2, \cdots, n$, and $g, h \in L_1(\mathbb{R}_+)$ are given, and f is an unknown.

Theorem

Let $\|g\|_1 < \dfrac{|\tan(\alpha_0\frac{\pi}{2})|k}{1+|\tan(\alpha_0\frac{\pi}{2})|}$. *Then the Caputo fractional integro-differential equation* (26) *has a unique solution* f *from* $BSA_{2(\alpha_0-\alpha_n)+3}(\mathbb{R}_+)$.

$$
\begin{aligned}
Q_j(s) &= \frac{s^{\alpha_j-1}}{s^{\alpha_0}+\sum_{i=1}^n a_i s^{\alpha_i}+k+G(s)}, \quad j=0,1,\cdots,n,\\
R_j(s) &= \frac{s^{\alpha_j-2}}{s^{\alpha_0}+\sum_{i=1}^n a_i s^{\alpha_i}+k+G(s)}, \quad j=0,1,\cdots,l, \quad (27)\\
f(t) &= f_0\, q_0(t)+f_1 r_0(t)+f_0\sum_{j=1}^n a_j q_j(t)\\
&+ f_1\sum_{j=1}^l a_j r_j(t)+\int_0^t m(t-\tau)h(\tau)\,d\tau.
\end{aligned}
$$

Mixed Caputo Riemann-Liouville Fractional Equation $\frac{1}{2} < \alpha_0 \leq 1$

Now we consider the following mixed Caputo Riemann-Liouville fractional integro-differential equation with a dominant Caputo fractional derivative

$$
\begin{gathered}
{}^{\mathcal{C}}\partial_t^{\alpha_0} f(t) + \sum_{j=1}^{n} a_j {}^{\mathcal{C}}\partial_t^{\alpha_j} f(t) + \sum_{j=1}^{m} b_j D_{0+}^{\beta_j} f(t) + kf(t) + \\
+ \int_0^t g(t-\tau) f(\tau) d\tau = h(t), \quad f(0+) = f_0, \quad \frac{1}{2} < \alpha_0 \leq 1, \\
0 < \alpha_n < \dots < \alpha_1 < \alpha_0, \quad 0 < \beta_m < \dots < \beta_1 < \alpha_0,
\end{gathered} \tag{28}
$$

where $g, h \in L_1(\mathbb{R}_+)$, $a_1, \cdots, a_n$, $b_1, \cdots, b_m$, $k \in \mathbb{R}_+$, are given, and f is an unknown.

Uniqueness in $BSA_{2(\alpha_0-\alpha_n)+1}(\mathbb{R}_+)$

Theorem

Let $k>0$, $f_0\in\mathbb{R}$, $g,h\in L_1(\mathbb{R}_+)$, be given, and $\|g\|_1<k$. Then the mixed Caputo Riemann-Liouville fractional integro-differential equation (28) has a unique solution f from $BSA_{2(\alpha_0-\alpha_n)+1}(\mathbb{R}_+)$.

Denote

$$L_j(s)=\frac{s^{\alpha_j-1}}{s^{\alpha_0}+\sum_{i=1}^n a_i s^{\alpha_i}+\sum_{i=1}^m b_i s^{\beta_i}+k+G(s)},\quad j=0,1,\cdots,n,$$

$$M(s)=\frac{1}{s^{\alpha_0}+\sum_{i=1}^n a_i s^{\alpha_i}+\sum_{i=1}^m b_i s^{\beta_i}+k+G(s)},\tag{29}$$

then

$$f(t)=f_0\, l_0(t)+f_0\sum_{j=1}^n a_j l_j(t)+\int_0^t m(t-\tau)h(\tau)\,d\tau.\tag{30}$$

Mixed Caputo Riemann-Liouville Fractional Equation $1 < \alpha_0 \le 2$

Consider the following mixed Caputo Riemann-Liouville fractional integro-differential equation with a dominant Caputo fractional derivative

$$
\begin{aligned}
& {}^{\mathcal{C}}\partial_t^{\alpha_0} f(t) + \sum_{j=1}^{n} a_j \, {}^{\mathcal{C}}\partial_t^{\alpha_j} f(t) + \sum_{j=1}^{m} b_j D_{0+}^{\beta_j} f(t) \\
+ \quad & k f(t) + \int_0^t g(t-\tau) f(\tau) d\tau = h(t), \; f(0+) = f_0, \; f'(0+) = f_1, \\
& 0 < \alpha_n < \cdots < \alpha_{l+1} < 1 < \alpha_l < \cdots < \alpha_1 < \alpha_0 \le 2, \\
& 0 < \beta_m < \cdots < \beta_1 < 1, \qquad (31)
\end{aligned}
$$

where $g, h \in L_1(\mathbb{R}_+), a_1, \cdots, a_n, \; b_1, \cdots, b_m, \; k \in \mathbb{R}_+$, are given, and f is an unknown.

Uniqueness in $BSA_{2(\alpha_0-\alpha_n)+3}(\mathbb{R}_+)$

Theorem

Let $f_0,\ f_1 \in \mathbb{R},\ \ g, h \in L_1(\mathbb{R}_+)$*, be given, and* $\|g\|_1 < \dfrac{|\tan(\alpha_0\frac{\pi}{2})|k}{1+|\tan(\alpha_0\frac{\pi}{2})|}$*. Then the mixed Caputo Riemann-Liouville fractional integro-differential equation* (31) *has a unique solution* f *from* $BSA_{2(\alpha_0-\alpha_n)+3}(\mathbb{R}_+)$*.*

$$\begin{aligned} X_j(s) &= \frac{s^{\alpha_j-1}}{s^{\alpha_0}+\sum_{i=1}^n a_i s^{\alpha_i}+\sum_{i=1}^m b_i s^{\beta_i}+k+G(s)}, \quad j=0,\cdots,n, \\ Y_j(s) &= \frac{s^{\alpha_j-2}}{s^{\alpha_0}+\sum_{i=1}^n a_i s^{\alpha_i}+\sum_{i=1}^m b_i s^{\beta_i}+k+G(s)}, \quad j=0,\cdots,l, \\ Z(s) &= \frac{1}{s^{\alpha_0}+\sum_{i=1}^n a_i s^{\alpha_i}+\sum_{i=1}^m b_i s^{\beta_i}+k+G(s)}, \qquad (32) \end{aligned}$$

$$\begin{aligned} f(t) &= f_0\, x_0(t)+f_1 y_0(t)+f_0\sum_{j=1}^n a_j x_j(t) \\ &+ f_1\sum_{j=1}^l a_j y_j(t)+\int_0^t z(t-\tau)h(\tau)\,d\tau \qquad (33) \end{aligned}$$

Partial Caputo Fractional Integro-Differential Equation $\frac{1}{2} < \alpha < 1$

$$\begin{cases} {}^{\mathcal{C}}\partial_t^{\alpha} u(x,t) = k\Delta u(x,t) - \int_0^t g(t-\tau)u(x,\tau)d\tau, & (x,t) \in \Omega \times \mathbb{R}^+, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = f(x), & x \in \Omega, \end{cases} \tag{34}$$

with $\frac{1}{2} < \alpha \le 1$, where $\Omega \subset \mathbb{R}^d$ $(d \ge 1)$ is a bounded domain with smooth boundary $\partial\Omega \in C^{\left[\frac{d}{2}\right]+1}$. Here $\left[\frac{d}{2}\right]$ denotes the integer part of $\frac{d}{2}$. The model in (34) appears in many modeling situations of new viscoelastic materials such as polymers.

Eigenvalues and Eigenfunctions

As we shall use spectral methods associated with the Dirichlet Laplacian, denote its eigenvalues indexed in the ascending order and counting their multiplicity, by λ_j and associate eigenfunctions by φ_j, i.e.

$$\begin{cases} \Delta\varphi_j(x) = -\lambda_j\varphi_j(x), & \text{in } \Omega, \\ \varphi_j(x) = 0, & \text{on } \partial\Omega. \end{cases} \tag{35}$$

It is known that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_j \leq \cdots$, with $\lim_{j\to\infty} \lambda_j = \infty$, and the set $\{\varphi_j\}_{j\geq 1}$, normalized by $||\varphi_j||_{L_2(\Omega)} = 1$, is an orthonormal basis for $L_2(\Omega)$. Moreover, $\varphi_j \in C^\infty(\Omega)$, and the smoothness condition on the boundary guarantees that $\varphi_j \in C(\overline{\Omega})$.

Particular Solutions

First we look for a particular solution of (50) in the form

$$u(x,t) = c_j(t)\varphi_j(x),$$

where $u(x,0) = \varphi_j(x)$, so c_j satisfies the Caputo fractional integro-differential equation,

$$^{\mathcal{C}}\partial_t^{\alpha} c_j(t) = -k\lambda_j c_j(t) - \int_0^t g(t-s)c_j(s)\,ds, \qquad \text{with} \qquad c_j(0) = 1. \tag{36}$$

Equation (36) has the unique solution

$$C_j(s) = \frac{s^{\alpha-1}}{s^{\alpha} + \lambda_j k + G(s)}, \quad c_j(t) = \left(\mathcal{L}^{-1} C_j\right)(s). \tag{37}$$

Theorem

Let $\frac{1}{2} < \alpha \leq 1$, $\|g\|_1 < \lambda_1 k$. Then $c_j(t)$, defined by (37), belongs to $BSA_1^1(\mathbb{R}_+)$.

Fourier Coefficients

By f_j we denote the j^{th} Fourier coefficient of $f \in L_2(\Omega)$ in the basis $\{\varphi_j\}_{j\geq 1}$,

$$f_j = \int_\Omega f(x)\varphi_j(x)\, dx.$$

If $f \in H_0^m(\Omega)$, the Sobolev space of functions with compact supports in Ω with generalized derivatives up to order $m \geq 0$, then its Fourier coefficient f_j has the asymptotics

$$f_j = O\left(j^{-\frac{m}{d}}\right), \quad j \to \infty. \tag{38}$$

The Classical Solution

Theorem

Let $g \in L_1(\mathbb{R}_+) \cup L_\infty(\mathbb{R}_+), f \in H_0^m(\Omega), \partial\Omega \in C^m$ *with* $m > \frac{3d+3}{2}, \frac{1}{2} < \alpha \leq 1,$ *and* $\|g\|_1 < k\lambda_1$*. Then* $u(x,t)$*, defined by*

$$u(x,t) := \sum_{j=1}^{\infty} f_j c_j(t) \varphi_j(x) \tag{39}$$

is the unique classical solution of (34) *in* $C^2(\overline{\Omega}) \times BSA_1^1(\mathbb{R}_+)$.

Partial Riemann-Liouville Fractional Integro-Differential Equation $\frac{1}{2} < \alpha \leq 1$

$$\begin{cases} D_{0+}^{\alpha} u(x,t) = k\Delta u(x,t) - \int_0^t g(t-\tau)u(x,\tau)d\tau, & (x,t) \in \Omega \times \mathbb{R}^+, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ I_{0+}^{1-\alpha} u(x,0) = f(x), & x \in \Omega, \end{cases} \tag{40}$$

with $\frac{1}{2} < \alpha \leq 1$, where $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega \in C^{[\frac{d}{2}]+1}$.

Particular Solutions

We look for a particular solution of (40) in the form

$$u(x,t) = c_j(t)\varphi_j(x), \tag{41}$$

where $I_{0+}^{1-\alpha}u(x,0) = \varphi_j(x)$, so $c_j(t)$ satisfies the fractional integro-differential equation,

$$D_{0+}^{\alpha}c_j(t) = -k\lambda_j c_j(t) - \int_0^t g(t-s)c_j(s)\,ds, \quad \text{with} \quad I_{0+}^{1-\alpha}c_j(0) = 1. \tag{42}$$

Equation (42) has the solution

$$C_j(s) = \frac{1}{s^{\alpha} + \lambda_j k + G(s)}, \quad c_j(t) = \left(\mathcal{L}^{-1}C_j\right)(s). \tag{43}$$

Theorem

Let $\frac{1}{2} < \alpha \leq 1$ and $\|g\|_1 < \lambda_1 k$. Then $c_j(t)$, defined by (43), belongs to $BSA_1(\mathbb{R}_+)$.

Theorem

Let $g \in L_1(\mathbb{R}_+) \cup L_\infty(\mathbb{R}_+), f \in H_0^m(\Omega), \partial\Omega \in C^m$ *with* $m > \frac{3d+3}{2}, \frac{1}{2} < \alpha \leq 1$, *and* $\|g\|_1 < k\lambda_1$. *Then* $u(x,t)$, *defined by*

$$u(x,t) := \sum_{j=1}^{\infty} f_j c_j(t) \varphi_j(x), \qquad (44)$$

is the unique classical solution of (40) *in* $C^2(\overline{\Omega}) \times BSA_1^\alpha(\mathbb{R}_+)$.

By $f(t) \in BSA_p^\alpha(\mathbb{R}_+)$ we mean both $f(t), D_{0+}^\alpha f(t) \in BSA_p(\mathbb{R}_+)$.

Partial Riemann-Liouville Fractional Integro-Differential Equation $\frac{3}{2} < \alpha < 2$

$$\begin{cases} D_{0+}^{\alpha} u(x,t) = k\Delta u(x,t) - \int_0^t g(t-\tau)u(x,\tau)d\tau, & (x,t) \in \Omega \times \mathbb{R}^+, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ I_{0+}^{2-\alpha} u(x,0) = f(x), \quad D_{0+}^{\alpha-1} u(x,0) = h(x), & x \in \Omega, \end{cases} \tag{45}$$

with $\frac{3}{2} < \alpha < 2$, where $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega \in C^{[\frac{d}{2}]+1}$.

Particular Solutions

First we look for a particular solution of (45) in the form

$$u(x,t) = [c_j(t) + d_j(t)]\varphi_j(x), \tag{46}$$

satisfying $I_{0+}^{2-\alpha}u(x,0) = D_{0+}^{\alpha-1}u(x,0) = \varphi_j(x)$.

$$C_j(s) = \frac{s}{s^\alpha + \lambda_j k + G(s)}, \quad c_j(t) = \left(\mathcal{L}^{-1}C_j\right)(s), \tag{47}$$

$$D_j(s) = \frac{1}{s^\alpha + \lambda_j k + G(s)}, \quad d_j(t) = \left(\mathcal{L}^{-1}D_j\right)(s). \tag{48}$$

Theorem

Let $\frac{3}{2} < \alpha < 2$, $\|g\|_1 < \dfrac{|\tan(\frac{\alpha\pi}{2})|}{1+|\tan(\frac{\alpha\pi}{2})|}k\lambda_1$. *Then* $c_j(t)$ *and* $d_j(t)$, *defined by* (47) *and* (48), *belong to* $BSA^\alpha_{2\alpha-3}(\mathbb{R}_+)$ *and* $BSA^\alpha_{2\alpha-1}(\mathbb{R}_+)$, *respectively.*

Theorem

Let $g \in L_1(\mathbb{R}_+) \cup L_\infty(\mathbb{R}_+)$, $f \in H_0^m(\Omega)$, $h \in H_0^{m-1}(\Omega)$, $\partial\Omega \in C^m$ *with* $m > \frac{3d+5}{2}$, $\frac{3}{2} < \alpha < 2$, *and* $\|g\|_1 < \dfrac{|\tan(\frac{\alpha\pi}{2})|}{1 + |\tan(\frac{\alpha\pi}{2})|} k\lambda_1$. *Then* $u(x,t)$, *defined by*

$$u(x,t) := \sum_{j=1}^{\infty} [f_j c_j(t) + h_j d_j(t)] \varphi_j(x), \tag{49}$$

where f_j, h_j *are the* j^{th} *Fourier coefficients of* $f, h \in L_2(\Omega)$ *in the basis* $\{\varphi_j\}_{j\geq 1}$, *namely* $\left\{ \begin{array}{c} f_j \\ h_j \end{array} \right\} = \int_\Omega \left\{ \begin{array}{c} f(x) \\ h(x) \end{array} \right\} \varphi_j(x)\, dx$, *is the unique classical solution of* (45) *in* $C^2(\overline{\Omega}) \times BSA^{\alpha}_{2\alpha-1}(\mathbb{R}_+)$.

Partial Caputo Fractional Integro-Differential Equation $\frac{3}{2} < \alpha < 2$

$$\begin{cases} {}^{\mathfrak{C}}\partial_t^{\alpha} u(x,t) = k\Delta u(x,t) - \int_0^t g(t-\tau)u(x,\tau)d\tau, & (x,t) \in \Omega \times \mathbb{R}^+, \\ u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = f(x), \quad \frac{\partial u(x,t)}{\partial t}\Big|_{t=0} = h(x), \qquad x \in \Omega, , & x \in \Omega, \end{cases} \tag{50}$$

with $\frac{3}{2} < \alpha < 2$, where $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega \in C^{[\frac{d}{2}]+1}$.

Particular Solutions

We again look for a particular solution of (50) in the form

$$u(x,t) = [c_j(t) + d_j(t)]\varphi_j(x), \tag{51}$$

but under the initial conditions $u(x,0) = \frac{\partial u(x,t)}{\partial t}|_{t=0} = \varphi_j(x)$. If $\|g\|_1 < \dfrac{|\tan(\frac{\alpha\pi}{2})|}{1+|\tan(\frac{\alpha\pi}{2})|} k\lambda_1$, $\frac{3}{2} < \alpha < 2$, then

$$c_j(t) = \left(\mathcal{L}^{-1}C_j\right)(s) \in BSA^2_{2\alpha-1}(\mathbb{R}_+), \quad C_j(s) = \frac{s^{\alpha-1}}{s^\alpha + \lambda_j k + G(s)} \tag{52}$$

and

$$d_j(t) = \left(\mathcal{L}^{-1}D_j\right)(s) \in BSA^2_3(\mathbb{R}_+), \quad D_j(s) = \frac{s^{\alpha-2}}{s^\alpha + \lambda_j k + G(s)}. \tag{53}$$

The Classical Solution

Theorem

Let $g \in L_1(\mathbb{R}_+) \cup L_\infty(\mathbb{R}_+)$, $f \in H_0^m(\Omega)$, $h \in H_0^{m-1}(\Omega)$, $\partial\Omega \in C^m$ *with* $m > \frac{3d+5}{2}$, $\frac{3}{2} < \alpha < 2$, *and* $\|g\|_1 < \dfrac{|\tan(\frac{\alpha\pi}{2})|}{1+|\tan(\frac{\alpha\pi}{2})|} k\lambda_1$. *Then* $u(x,t)$, *defined by*

$$u(x,t) := \sum_{j=1}^{\infty} [f_j c_j(t) + h_j d_j(t)] \varphi_j(x), \tag{54}$$

where f_j, h_j *are the* j^{th} *Fourier coefficients of* $f, h \in L_2(\Omega)$ *in the basis* $\{\varphi_j\}_{j\geq 1}$, *is the unique classical solution of* (50) *in* $C^3(\overline{\Omega}) \times BSA_3^2(\mathbb{R}_+)$.

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THANK YOU