On the theory of the subtrigonometric functions

Xiao-Jun Yang

Email: dyangxiaojun@163.com
School of Mathematics,
and State Key Laboratory for Geo-Mechanics and Deep Underground Engineering,
China University of Mining and Technology Xuzhou 221116, Jiangsu, P. R. China,
and Department of Mathematics,
Faculty of Science, King Abdulaziz University, P.O. Box 80257 Jeddah 21589,
Saudi Arabia

International Biweekly Online Seminar on Analysis, Differential Equations AND Mathematical Physics
Regional Mathematical Center, Southern Federal University, Russia, 29 September 2022

Abstract

The lecture is devoted to all zeros of the special functions in the Wiman class. The ordinary differential equations for them are obtained in detail. Finally, we give the comments for open problems in the theory of the subtrigonometric functions.

Contents

The background of the problems

Let $\mathbb{C}$ and $\mathbb{N}$ denote the sets of the complex numbers and natural numbers, respectively. Consider that $\widehat{\mathbb{N}} = \mathbb{N} \cup \{0\}$ such that $k \in \widehat{\mathbb{N}}$. The exponential function is expressed as [1]

$$e^t = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k+1)}, \tag{1}$$

where $t \in \mathbb{C}$, and Γ denotes the gamma function. Let $Re(t)$ be the real part of a complex variable $t \in \mathbb{C}$. The well-known Mittag-Leffler function, proposed in 1903 by Mittag-Leffler [2, 3], is given as (see [2]; for its investigation, see [3])

$$\widehat{E}_\theta(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\theta k+1)}, \tag{2}$$

where $t, \theta \in \mathbb{C}$ and $Re(\theta) > 0$.

After two years, as the extended version of the Mittag-Leffler function (2), the well-known Wiman function, suggested in 1905 by Wiman [4], is represented in the form (see [4]; for its history, see [3]):

$$\widehat{E}_{\theta,\rho}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\theta k + \rho)}, \tag{3}$$

where $t, \theta, \rho \in \mathbb{C}$, $Re(\theta) > 0$, and $Re(\rho) > 0$.
Recently, the subsine functions of first type, introduced by me and my students in [5], is defined as [5]

$$ {}_1subsin_2(t) := \sum_{k=0}^{\infty} (-1)^k \frac{t^{4k+1}}{\Gamma(4k+2)}, \tag{4}$$

and

$$ {}_1subsin_4(t) := \sum_{k=0}^{\infty} (-1)^k \frac{t^{4k+3}}{\Gamma(4k+4)}, \tag{5}$$

and the subcosine functions of first type as [5]

$$ {}_1subcos_1\left(t\right) := \sum_{k=0}^{\infty} \left(-1\right)^k \frac{t^{4k}}{\Gamma\left(4k+1\right)}, \tag{6} $$

and

$$ {}_1subcos_3\left(t\right) := \sum_{k=0}^{\infty} \left(-1\right)^k \frac{t^{4k+2}}{\Gamma\left(4k+3\right)}. \tag{7} $$

The plots of the special functions ${}_1subsin_2\left(t\right)$, ${}_1subsin_4\left(t\right)$, ${}_1subcos_1\left(t\right)$ and ${}_1subcos_3\left(t\right)$ are shown in Figures 1, 2, 3, and 4, respectively.

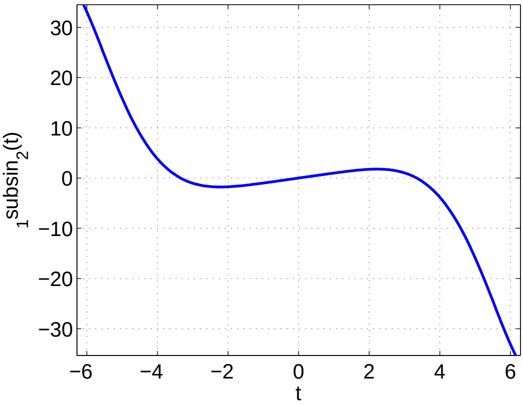

Figure 1: The plot of the subsine functions of first type ${}_1 subsin_2\,(t)$ in the real domain.

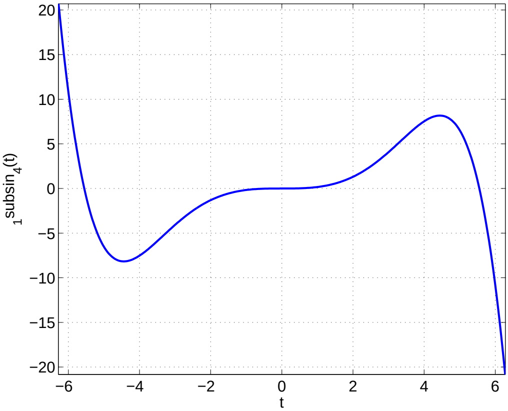

Figure 2: The plot of the subsine functions of first type $_1subsin_4(t)$ in the real domain.

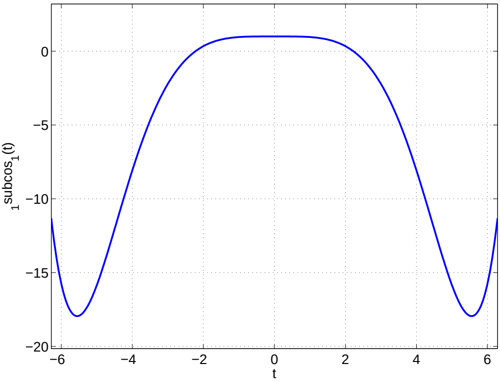

Figure 3: The plot of the subsine functions of first type ${}_1subcos_1(t)$ in the real domain.

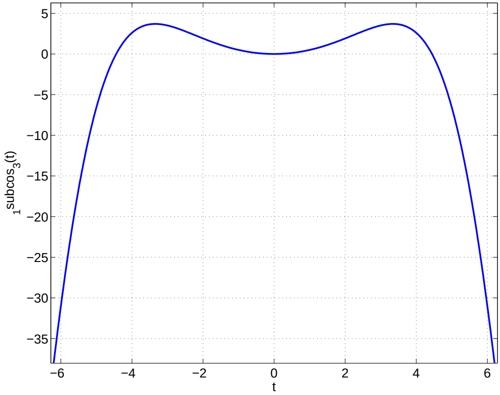

Figure 4: The plot of the subcosine functions of first type $_{1}subcos_{3}\left(t\right)$ in the real domain.

The imaginary unit i is expressed as [1]

$$i := \sqrt{-1}. \tag{8}$$

We now introduce the unit of the number λ by [5]

$$\lambda := \sqrt{i}. \tag{9}$$

It is reported that ([6], p.191)

$$\lambda = \sqrt[4]{-1}. \tag{10}$$

It is easy to show that

$$_1 subsin_2 (t) = t\widehat{E}_{4,2} (\lambda t), \tag{11}$$

$$_1 subsin_4 (t) = t^3\widehat{E}_{4,4} (\lambda t), \tag{12}$$

$$_1 subcos_1 (t) = \widehat{E}_{4,1} (\lambda t), \tag{13}$$

and

$$_1 subcos_3 (t) = t^2\widehat{E}_{4,3} (\lambda t). \tag{14}$$

Thus, they are considered in the Wiman class [5].

The statement of the problems

Based on the representations of the special functions (4), (5), (6) and (7), we now introduce the conjectures as follows [5]:

Conjecture 1

Let $t \in \mathbb{C}$. Then all of zeros of the equation

$$_1subcos_1\left(t\right) = 0 \tag{15}$$

are real.

Conjecture 2

Let $t \in \mathbb{C}$. Then all of zeros of the equation

$$_1subsin_2\left(t\right) = 0 \tag{16}$$

are real.

Conjecture 3

Let $t \in \mathbb{C}$. Then all of zeros of the equation

$$_{1}subcos_{3}\left(t\right) = 0 \tag{17}$$

are real.

Conjecture 4

Let $t \in \mathbb{C}$. Then all of zeros of the equation

$$_{1}subsin_{4}\left(t\right) = 0 \tag{18}$$

are real.

The target of the present article

The main targets of the article are given as follows. In Section 2, we discuss the relationships among the sine, cosine, subsine and subcosine functions. In Section 3, we present the proofs of Conjectures 1 and 3. In Section 4, we consider the ordinary differential equations (ODEs) of the subsine and subcosine functions. In Section 5, we give the comment on the open problems in the theory of the subtrigonometric functions.

Following Euler [7], we have that (see [8], p.75)

$$\sin(t) = t\prod_{\ell=1}^{\infty}\left(1-\frac{t^2}{\ell^2\pi^2}\right) \tag{19}$$

and (see [8], p.75)

$$\cos(t) = \prod_{\ell=1}^{\infty}\left(1-\frac{4t^2}{(2\ell-1)^2\pi^2}\right). \tag{20}$$

Let us consider [1]

$$\sinh(t) = -i\sin(it), \tag{21}$$

and [1]

$$\cos(t) = \cosh(it). \tag{22}$$

Then we have

$$\sum_{k=0}^{\infty}\frac{t^{4k}}{\Gamma(4k+1)} = \frac{1}{2}\left(\cosh(t)+\cos(t)\right). \tag{23}$$

Moreover, we obtain

$$\sum_{k=0}^{\infty} \frac{t^{4k+2}}{\Gamma(4k+3)} = \frac{1}{2}\left(\cosh(t) - \cos(t)\right), \tag{24}$$

$$\sum_{k=0}^{\infty} \frac{t^{4k+1}}{\Gamma(4k+2)} = \frac{1}{2}\left(\sinh(t) + \sin(t)\right), \tag{25}$$

and

$$\sum_{k=0}^{\infty} \frac{t^{4k+3}}{\Gamma(4k+4)} = \frac{1}{2}\left(\sinh(t) - \sin(t)\right). \tag{26}$$

In a similar way, we may have

$${}_1subsin_2(t) = \frac{1}{2\lambda}\left(\sinh(\lambda t) + \sin(\lambda t)\right), \tag{27}$$

$${}_1subsin_4(t) = \frac{1}{2\lambda^3}\left(\sinh(\lambda t) - \sin(\lambda t)\right), \tag{28}$$

$${}_1subcos_1(t) = \frac{1}{2}\left(\cosh(\lambda t) + \cos(\lambda t)\right), \tag{29}$$

and

$${}_1subcos_3(t) = \frac{1}{2\lambda^2}\left(\cosh(\lambda t) - \cos(\lambda t)\right). \tag{30}$$

Let us denote [5]

$$ {}_1subsinh_2\left(t\right) := \sum_{k=0}^{\infty} \frac{t^{4k+1}}{\Gamma\left(4k+2\right)}, \tag{31} $$

$$ {}_1subsinh_4\left(t\right) := \sum_{k=0}^{\infty} \frac{t^{4k+3}}{\Gamma\left(4k+4\right)}, \tag{32} $$

$$ {}_1subcosh_1\left(t\right) := \sum_{k=0}^{\infty} \frac{t^{4k}}{\Gamma\left(4k+1\right)}, \tag{33} $$

and

$$ {}_1subcosh_3\left(t\right) := \sum_{k=0}^{\infty} \frac{t^{4k+2}}{\Gamma\left(4k+3\right)}. \tag{34} $$

So, we present

$$ {}_1subsin_2(t) = \frac{1}{\lambda}\,{}_1subsinh_2(\lambda t), \tag{35} $$

$$ {}_1subsin_4(t) = \frac{1}{\lambda^3}\,{}_1subsin_4(\lambda t), \tag{36} $$

$$ {}_1subcos_1(t) = {}_1subcosh_1(\lambda t), \tag{37} $$

and

$$ {}_1subcos_3(t) = \frac{1}{\lambda^2}\,{}_1subcos_3(\lambda t). \tag{38} $$

All zeros of the function ${}_1subcos_1(t)$

In this section, all zeros of the special functions ${}_1subcos_1(t)$ and ${}_1subcos_3(t)$ are investigated due to the work of Euler [7].
By using (37), it is not difficult to present

$${}_1subcos_1(t) = \cos\left(\frac{(\lambda^2+1)\lambda}{2}t\right)\cos\left(\frac{(\lambda^2-1)\lambda}{2}t\right) = \prod_{\ell=1}^{\infty}\left(1-\frac{4t^4}{\pi^4(2\ell-1)^4}\right). \tag{39}$$

By (39), we take

$$1-\frac{4t^4}{\pi^4(2\ell-1)^4}=0. \tag{40}$$

It is easy to find from (40) that

$$t_1 = \alpha(2\ell-1) \tag{41}$$

and

$$t_2 = \beta(2\ell-1), \tag{42}$$

where

$$\alpha = \frac{\sqrt{2}}{2}\pi \tag{43}$$

and

$$\beta = i\frac{\sqrt{2}}{2}\pi. \tag{44}$$

From (29), we find

$${}_1subcos_1(t) = {}_1subcos_1(-t) \tag{45}$$

such that

$$t_3 = \overline{\alpha}(2\ell - 1) \tag{46}$$

and

$$t_4 = \overline{\beta}(2\ell - 1), \tag{47}$$

where

$$\overline{\alpha} = -\frac{\sqrt{2}}{2}\pi \tag{48}$$

and

$$\overline{\beta} = -i\frac{\sqrt{2}}{2}\pi. \tag{49}$$

It is clear that $t_1 = \alpha(2\ell - 1)$, $t_2 = \beta(2\ell - 1)$, $t_3 = \overline{\alpha}(2\ell - 1)$ and $t_4 = \overline{\beta}(2\ell - 1)$ are all zeros of ${}_1subcos_1(t)$.
Thus, Conjecture 1 is false.

Remark 1

Both $t_1 = \alpha(2\ell - 1)$ and $t_3 = \overline{\alpha}(2\ell - 1)$ are the points which has been observed in Figure 1.

All zeros of the function ${}_1subcos_3\left(t\right)$

From (38), we may arrive at

$$ {}_1subcos_3\left(t\right) = -\frac{1}{\lambda^2}\sin\left(\frac{\left(\lambda^2+1\right)\lambda}{2}t\right)\sin\left(\frac{\left(\lambda^2-1\right)\lambda}{2}t\right) = \frac{t^2}{2}\prod_{\ell=1}^{\infty}\left(1-\frac{t^4}{4\ell^4\pi^4}\right). \tag{50} $$

From (50), we obtain

$$ 1-\frac{t^4}{4\ell^4\pi^4}=0, \tag{51} $$

which reduces to

$$ \bar{t}_1 = \gamma\ell \tag{52} $$

and

$$ \bar{t}_2 = \varphi\ell, \tag{53} $$

where

$$ \gamma = \sqrt{2}\pi \tag{54} $$

and

$$ \varphi = i\sqrt{2}\pi. \tag{55} $$

In a similar manner, we have

$$ {}_1subcos_3\left(-t\right) = {}_1subcos_3\left(t\right) \tag{56} $$

such that

$$ \overline{t}_3 = \overline{\gamma}\ell \tag{57} $$

and

$$ \overline{t}_4 = \overline{\varphi}\ell, \tag{58} $$

where

$$ \overline{\gamma} = -\sqrt{2}\pi \tag{59} $$

and

$$ \overline{\varphi} = -i\sqrt{2}\pi. \tag{60} $$

From (50), (52), (53), (57) and (58), $\overline{t}_0 = 0$, $\overline{t}_1$, $\overline{t}_2$, $\overline{t}_3$, $\overline{t}_4$ are all zeros of ${}_1subcos_3\left(t\right)$.
Thus, Conjecture 3 is false.

Remark 2

Both $\overline{t}_1 = \gamma\ell$ and $\overline{t}_3 = \overline{\gamma}\ell$ have been observed in Figure 2.

From (35) we consider [5]

$$\frac{d}{dt}\left[{}_1subsin_2\left(t\right)\right]=\frac{1}{2}\left(\cosh\left(\lambda t\right)+\cos\left(\lambda t\right)\right), \tag{61}$$

$$\frac{d^2}{dt^2}\left[{}_1subsin_2\left(t\right)\right]=\frac{\lambda}{2}\left(\sinh\left(\lambda t\right)-\sin\left(\lambda t\right)\right), \tag{62}$$

$$\frac{d^3}{dt^3}\left[{}_1subsin_2\left(t\right)\right]=\frac{\lambda^2}{2}\left(\cosh\left(\lambda t\right)-\cos\left(\lambda t\right)\right), \tag{63}$$

and

$$\frac{d^4}{dt^4}\left[{}_1subsin_2\left(t\right)\right]=-\frac{1}{2\lambda}\left(\sinh\left(\lambda t\right)+\sin\left(\lambda t\right)\right), \tag{64}$$

which lead to

$$\frac{d^4}{dt^4}\Phi\left(t\right)+\Phi\left(t\right)=0, \tag{65}$$

where

$$\Phi\left(0\right)=0, \Phi^{(1)}\left(0\right)=1, \Phi^{(2)}\left(0\right)=0, \Phi^{(3)}\left(0\right)=0. \tag{66}$$

By (36), we may have

$$\frac{d}{dt}\left[{}_1subsin_4\left(t\right)\right]=\frac{1}{2\lambda^2}\left(\cosh\left(\lambda t\right)-\cos\left(\lambda t\right)\right), \tag{67}$$

$$\frac{d^2}{dt^2}\left[{}_1subsin_4\left(t\right)\right]=\frac{1}{2\lambda}\left(\sinh\left(\lambda t\right)+\sin\left(\lambda t\right)\right), \tag{68}$$

$$\frac{d^3}{dt^3}\left[{}_1subsin_4\left(t\right)\right]=\frac{1}{2}\left(\cosh\left(\lambda t\right)+\cos\left(\lambda t\right)\right), \tag{69}$$

and

$$\frac{d^4}{dt^4}\left[{}_1subsin_4\left(t\right)\right]=-\frac{1}{2\lambda^3}\left(\sinh\left(\lambda t\right)-\sin\left(\lambda t\right)\right), \tag{70}$$

which yield that

$$\frac{d^4}{dt^4}\Phi\left(t\right)+\Phi\left(t\right)=0, \tag{71}$$

where

$$\Phi\left(0\right)=0,\Phi^{(1)}\left(0\right)=0,\Phi^{(2)}\left(0\right)=0,\Phi^{(3)}\left(0\right)=1. \tag{72}$$

With (37), we can obtain

$$\frac{d}{dt}\left[{}_1subcos_1(t)\right]=\frac{\lambda}{2}\left(\sinh(\lambda t)-\sin(\lambda t)\right), \tag{73}$$

$$\frac{d^2}{dt^2}\left[{}_1subcos_1(t)\right]=\frac{\lambda^2}{2}\left(\cosh(\lambda t)-\cos(\lambda t)\right), \tag{74}$$

$$\frac{d^3}{dt^3}\left[{}_1subcos_1(t)\right]=\frac{\lambda^3}{2}\left(\sinh(\lambda t)+\sin(\lambda t)\right), \tag{75}$$

and

$$\frac{d^4}{dt^4}\left[{}_1subcos_1(t)\right]=-\left(\cosh(\lambda t)+\cos(\lambda t)\right), \tag{76}$$

which reduce to

$$\frac{d^4}{dt^4}\Phi(t)+\Phi(t)=0, \tag{77}$$

where

$$\Phi(0)=1, \Phi^{(1)}(0)=0, \Phi^{(2)}(0)=0, \Phi^{(3)}(0)=0. \tag{78}$$

In a similar way, it follows from (38) that

$$\frac{d}{dt}\left[{}_1subcos_3\left(t\right)\right]=\frac{1}{2\lambda}\left(\sinh\left(\lambda t\right)+\sin\left(\lambda t\right)\right), \tag{79}$$

$$\frac{d^2}{dt^2}\left[{}_1subcos_3\left(t\right)\right]=\frac{1}{2}\left(\cosh\left(\lambda t\right)+\cos\left(\lambda t\right)\right), \tag{80}$$

$$\frac{d^3}{dt^3}\left[{}_1subcos_3\left(t\right)\right]=\frac{\lambda}{2}\left(\sinh\left(\lambda t\right)+\sin\left(\lambda t\right)\right), \tag{81}$$

and

$$\frac{d^4}{dt^4}\left[{}_1subcos_3\left(t\right)\right]=-\frac{1}{2\lambda^2}\left(\cosh\left(\lambda t\right)-\cos\left(\lambda t\right)\right). \tag{82}$$

It follows from (7) that

$$\frac{d^4}{dt^4}\Phi\left(t\right)+\Phi\left(t\right)=0, \tag{83}$$

where

$$\Phi\left(0\right)=0,\Phi^{(1)}\left(0\right)=0,\Phi^{(2)}\left(0\right)=1,\Phi^{(3)}\left(0\right)=0. \tag{84}$$

By (6) and (30), we may show

$$\sum_{k=0}^{\infty}(-1)^k \frac{t^{4k}}{\Gamma(4k+1)} = \prod_{\ell=1}^{\infty}\left(1 - \frac{4t^4}{\pi^4(2\ell-1)^4}\right). \tag{85}$$

It is seen that (85) belongs to the special class since the right and left sides of (85) are the Taylor series and infinite product of ${}_1subcos_1(t)$, respectively.
In a similar way, from (7) and (50), we have that

$$\sum_{k=0}^{\infty}(-1)^k \frac{t^{4k+2}}{\Gamma(4k+3)} = \frac{t^2}{2}\prod_{\ell=1}^{\infty}\left(1 - \frac{t^4}{4\ell^4\pi^4}\right). \tag{86}$$

Here, (86) is also in the special class since the right and left sides of (86) are the Taylor series and infinite product of ${}_1subcos_3(t)$, respectively.

Applying (85) and (86), we get the followings:

$$\widehat{E}_{4,1}(t) = \prod_{\ell=1}^{\infty}\left(1+\frac{4t^4}{\pi^4(2\ell-1)^4}\right), \tag{87}$$

and

$$\widehat{E}_{4,3}(t) = \frac{1}{2}\prod_{\ell=1}^{\infty}\left(1+\frac{t^4}{4\ell^4\pi^4}\right). \tag{88}$$

Let us consider

$$\widehat{E}_{4,2}(\lambda t) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{4k}}{\Gamma(4k+2)} \tag{89}$$

and

$$\widehat{E}_{4,4}(\lambda t) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{4k}}{\Gamma(4k+4)}. \tag{90}$$

Thus, we may obtain an alternative representation of Conjecture 2 as follows:

Conjecture 5

Let $t \in \mathbb{C}$. Then all of zeros of the equation

$$\widehat{E}_{4,2}(\lambda t) = 0 \tag{91}$$

are real.

Similarly, we further show an alternative representation of Conjecture 4 as follows:

Conjecture 6

Let $t \in \mathbb{C}$. Then all of zeros of the equation

$$\widehat{E}_{4,4}(\lambda t) = 0 \tag{92}$$

are real.

Let us denote the presupercosine function $precos_{\theta,\rho}(t)$ by (see [3], p.392)

$$precos_{\theta,\rho}(t) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{\Gamma(2\theta k + \rho)}, \tag{93}$$

where $t, \theta, \rho \in \mathbb{C}$, $Re(\theta) > 0$, and $Re(\rho) > 0$.
It implies from (11), (12), (13), (14) and (93) that

$$precos_{2,1}\left(t^2\right)=\widehat{E}_{4,1}\left(\lambda t\right), \tag{94}$$

$$precos_{2,2}\left(t^2\right)=\widehat{E}_{4,2}\left(\lambda t\right), \tag{95}$$

$$precos_{2,3}\left(t^2\right)=\widehat{E}_{4,3}\left(\lambda t\right), \tag{96}$$

and

$$precos_{2,4}\left(t^2\right)=\widehat{E}_{4,4}\left(\lambda t\right). \tag{97}$$

Thus, we have the followings:

$$precos_{2,1}\left(t\right)=\prod_{\ell=1}^{\infty}\left(1-\frac{4t^2}{\pi^4\left(2\ell-1\right)^4}\right) \tag{98}$$

and

$$precos_{2,3}\left(t\right)=\frac{1}{2}\prod_{\ell=1}^{\infty}\left(1-\frac{t^2}{4\ell^4\pi^4}\right). \tag{99}$$

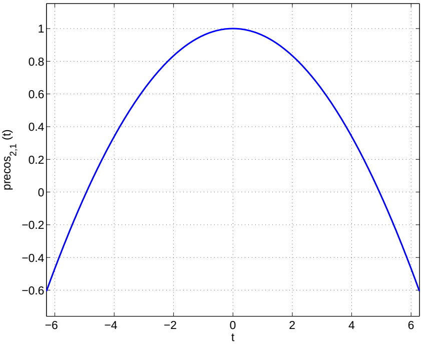

Figure 5: The plot of $precos_{2,1}(t) = \sum_{k=0}^{\infty} (-1)^k t^{4k} / \Gamma(4k+1)$ in the real domain.

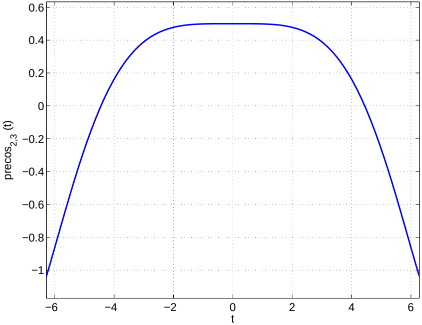

Figure 6: The plot of $precos_{2,3}(t) = \sum_{k=0}^{\infty} (-1)^k t^{4k} / \Gamma(4k+3)$ in the real domain.

Conjecture 7

Let $t \in \mathbb{C}$. Then all of zeros of the equation

$$presin_{2,2}\left(t\right) = 0 \tag{100}$$

are real.

Conjecture 8

Let $t \in \mathbb{C}$. Then all of zeros of the equation

$$presin_{2,4}\left(t\right) = 0 \tag{101}$$

are real.

Remark 3

It is easy to see that Conjectures 2 and 5 hold if Conjecture 7 is true, and that Conjectures 4 and 6 hold if Conjecture 8.

Combining (4), (5), (6), and (7), one denotes [5]

$$\tan\begin{bmatrix}(1,2)\\(1,1)\end{bmatrix}(t)=\frac{{}_1subsin_2(t)}{{}_1subcos_1(t)}\left({}_1subcos_1(t)\neq 0\right), \tag{102}$$

$$\cot\begin{bmatrix}(1,1)\\(1,2)\end{bmatrix}(t)=\frac{{}_1subcos_1(t)}{{}_1subsin_2(t)}\left({}_1subsin_2(t)\neq 0\right), \tag{103}$$

$$\sec\begin{bmatrix}(0,0)\\(1,1)\end{bmatrix}(t)=\frac{1}{{}_1subcos_1(t)}\left({}_1subcos_1(t)\neq 0\right), \tag{104}$$

$$\csc\begin{bmatrix}(0,0)\\(1,2)\end{bmatrix}(t)=\frac{1}{{}_1subsin_2(t)}\left({}_1subsin_2(t)\neq 0\right), \tag{105}$$

$$\tan\begin{bmatrix}(1,4)\\(1,3)\end{bmatrix}(t)=\frac{{}_1subsin_4(t)}{{}_1subcos_3(t)}\left({}_1subcos_3(t)\neq 0\right), \tag{106}$$

$$\cot\begin{bmatrix}(1,3)\\(1,4)\end{bmatrix}(t)=\frac{{}_1subcos_3(t)}{{}_1subsin_4(t)}\left({}_1subsin_4(t)\neq 0\right), \tag{107}$$

$$\sec\begin{bmatrix}(0,0)\\(1,3)\end{bmatrix}(t)=\frac{1}{{}_1subcos_3(t)}\left({}_1subcos_3(t)\neq 0\right), \tag{108}$$

and

$$\csc\begin{bmatrix}(0,0)\\(1,4)\end{bmatrix}(t)=\frac{1}{{}_1subsin_4(t)}\left({}_1subsin_4(t)\neq 0\right). \tag{109}$$

The above formulas are considered as the theory of the subtrigonometric functions [5].

Thus, one reconsiders the representations of (102), (104), (106) and (108) as

$$\tan\begin{bmatrix}(1,2)\\(1,1)\end{bmatrix}(t)=\frac{{}_1subsin_2(t)}{{}_1subcos_1(t)}, \tag{110}$$

where $t\neq\pm\frac{\sqrt{2}}{2}\pi(2\ell-1)$ and $\pm i\frac{\sqrt{2}}{2}\pi(2\ell-1)$,

$$\sec\begin{bmatrix}(0,0)\\(1,1)\end{bmatrix}(t)=\frac{1}{{}_1subcos_1(t)}, \tag{111}$$

where $t\neq\pm\frac{\sqrt{2}}{2}\pi(2\ell-1)$ and $\pm i\frac{\sqrt{2}}{2}\pi(2\ell-1)$,

$$\tan\begin{bmatrix}(1,4)\\(1,3)\end{bmatrix}(t)=\frac{{}_1subsin_4(t)}{{}_1subcos_3(t)}, \tag{112}$$

where $t\neq 0,\ \pm\sqrt{2}\pi\ell$ and $\pm i\sqrt{2}\pi\ell$, and

$$\sec\begin{bmatrix}(0,0)\\(1,3)\end{bmatrix}(t)=\frac{1}{{}_1subcos_3(t)}, \tag{113}$$

where $t\neq 0, \pm\sqrt{2}\pi\ell$ and $\pm i\sqrt{2}\pi\ell$.
Taking $t=\frac{\pi\lambda}{2}$ in (50), we show

$$2\left(e^{\frac{\pi}{2}}+e^{-\frac{\pi}{2}}\right)=\pi^2\prod_{\ell=1}^{\infty}\left(1+\frac{1}{64\ell^4}\right). \tag{114}$$

It is easy to see that (114) is a useful formula connected with π.

Remark 4

The preperties of λ are given as follows: $\lambda^2=i$, $\lambda^3=i\lambda$, $\lambda^4=-1$, $\lambda^5=-\lambda$, $\lambda^6=-i$, $\lambda^7=-i\lambda$, and $\lambda^8=1$.

Thank you for your attention!

- Knopp, K. (2016). Elements of the theory of functions. Courier Dover Publications.
- Mittag-Leffler, G. M. (1903). Sur la nouvelle fonction $E_\alpha(x)$. Comptes Rendus de l'Académie des Sciences, 137, 554–558.
- Yang, X. J. (2021). Theory and applications of special functions for scientists and engineers. Springer Singapore.
- Wiman, A. (1905). Über den fundamentalsatz in der teorie der funktionen $E_\alpha(x)$. Acta Mathematica, 29, 191–201.
- Yang, X.-J., Geng, L.-L., and Fan. Y.-R. (2021). New special functions applied to represent the Weierstrass-Mandelbrot function, Fractals, accepted.
- Cajori, F. (1993). A history of mathematical notations (Vol. 1). Courier Corporation.

- Euler, L. (1748). Introductio in analysin infinitorum (Vol. 1). Apud Marcum-Michaelem Bousquet & Socios.
- Abramowitz, M., and Stegun, I. A. (Eds.). (1964). Handbook of mathematical functions with formulas, graphs, and mathematical tables (Vol. 55). US Government printing office.