Integrability properties of integral transforms via Morrey spaces

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Introduction

This lecture is based on recent results of the paper [N.Samko. Integrability properties of integral transforms via Morrey spaces. *Fract.Calc.Appl.Anal.*, 2020 (to appear)].

We show that weighted local Morrey spaces and the so called complementary Morrey spaces provide a very natural language for describing integrability properties of integral transforms

$$Af(x) = \int_0^\infty k(xt)f(t)dt, \ x > 0, \tag{1}$$

with kernel depending on the product of arguments, in particular, Laplace transform

$$\mathcal{L}f(x) = \int_0^\infty e^{-xt}f(t)dt, \ x > 0, \tag{2}$$

Operators of the form (1), besides the Laplace transform, include in particular various forms of Bessel transform, Mittag-Leffler transform and others. Many concrete transforms of the form (1) are particular cases of so called H-transforms.

As is well known, behavior of integral transforms Af at infinity is interrelated with behavior of the original function f at the origin, which formally is seen from the equality

$$Af(x) = \frac{1}{x} \int_0^\infty k(t) f\left(\frac{t}{x}\right) dt.$$

To describe this interrelation we need an appropriate choice of the domain and target space for the transform. A first natural candidate is

Lebesgue space $L^p(\mathbb{R}_+)$ for the domain space,
[Hardy, THE CONSTANTS OF CERTAIN INEQUALITIES, 1933].

In the case of Laplace transform Hardy showed that the mapping

$\mathcal{L} : L^p(\mathbb{R}_+) \hookrightarrow L^q(\mathbb{R}_+)$ holds if and only if $1 \le p \le 2$ and $q = p'$.

The necessity of the condition $q = p'$ is well known to be obtained by delation arguments.

The next natural candidate is

L^p space with power weight.

In the same paper G.H.Hardy also proved the one-weight

$L^p \to L^p$ boundedness of the general operator A in the form

$$\int_0^\infty |Af(x)|^p dx \le c \int_0^\infty |f(x)|^p x^{p-2} dx$$

and

$$\int_0^\infty |Af(x)|^p x^{p-2} dx \le c \int_0^\infty |f(x)|^p dx$$

under some condition on the kernel $k(x)$ (necessary and sufficient when $k(x) \ge 0$).

More general weighted $L^p(u) \to L^q(v)$-setting:
[K.F. Andersen and H.P. Heinig. Weighted norm inequalities for certain integral operators, 1983] and [S. Bloom. Hardy integral estimates for the Laplace transform, 1992],

Rearrangement invariant spaces: [K.F. Andersen. On Hardyr's inequality and Laplace transforms in weighted rearrangement invariant spaces, 1973].

Since the Laplace transform is well defined on appropriate functions f when x is replaced by $z = x + iy$ with $x > 0$, integrability of Laplace transforms of L^p functions was also studied on the half plane $\{(x, y) : x > 0\}$, [S. Saitoh. The Laplace transform of L^p functions with weights, 1986].

In the real value setting, within the frameworks of rearrangement invariant function spaces there was achieved a progress in characterizing the best possible domain-target candidates, [E. Buriánková, D.E. Edmunds, and L. Pick. Optimal function spaces for the Laplace transform, 2017] and [O. Galdames Bravo. On the optimal domain of the Laplace transform, 2017].

The main results

We show that for any operator A of the form (1), under certain conditions on its kernel $k(t)$, there hold the inequalities

$$\sup_{r>0} r^{\lambda} \int_r^{\infty} x^b |Af(x)|^p dx \leq C \sup_{r>0} \frac{1}{r^{\lambda}} \int_0^r x^a |f(x)|^p dx, \quad \lambda \geq 0, \tag{3}$$

$$\sup_{r>0} \frac{1}{r^{\lambda}} \int_0^r x^b |Af(x)|^p dx \leq C \sup_{r>0} r^{\lambda} \int_r^{\infty} x^a |f(x)|^p dx, \quad \lambda \geq 0, \tag{4}$$

where $a, b \in \mathbb{R}$ are related to each other by the (necessary and sufficient) condition

$$a + b = p - 2;$$

for non-negative kernels $k(t)$ we also find sharp constants in the above inequalities.

For the corresponding vanishing Morrey and complementary Morrey spaces:

$$\lim_{r\to 0} \frac{1}{r^\lambda} \int_0^r x^a |f(x)|^p dx = 0 \Longrightarrow \lim_{r\to\infty} r^\lambda \int_r^\infty x^b |Af(x)|^p dx = 0, \quad \lambda > 0 \tag{5}$$

and

$$\lim_{r\to\infty} r^\lambda \int_r^\infty x^a |f(x)|^p dx = 0 \Longrightarrow \lim_{r\to 0} \frac{1}{r^\lambda} \int_0^r x^b |Af(x)|^p dx = 0, \quad \lambda > 0, \tag{6}$$

under the same conditions on the kernel.

The proofs are based on our results on integral operators with homogeneous kernel, in Morrey spaces, obtained in [N.Samko. Integral operators commuting with dilations and rotations in generalized Morrey-type spaces, 2020].

Weighted local Morrey and complementary Morrey spaces

For a function f on $\mathbb{R}^n$ we introduce the notation for the following modular

$$\mathfrak{M}_0^{p,\lambda,\gamma}(f,r) := \frac{1}{r^\lambda} \int\limits_{|x|<r} (|f(x)||x|^\gamma)^p \, dx, \tag{7}$$

where $1 \le p < \infty$, $\lambda \ge 0$ and $\gamma \in \mathbb{R}$.

Weighted local Morrey spaces $L^{p,\lambda,\gamma}(\mathbb{R}^n)$ are defined by the norm

$$\|f\|_{L^{p,\lambda,\gamma}} = \sup_{r>0} \left(\mathfrak{M}_0^{p,\lambda,\gamma}(f,r)\right)^{\frac{1}{p}}.$$

Recall that

$$L^{p,\lambda,\gamma}(\mathbb{R}^n)\big|_{\lambda=0} = L^{p,\gamma}(\mathbb{R}^n) =: \Big\{ \int\limits_{\mathbb{R}^n} |f(x)|x|^\gamma|^p dx < \infty \Big\}.$$

Vanishing weighted local Morrey space $V_0L^{p,\lambda,\gamma}(\mathbb{R}^n),\ \lambda > 0$:

$$\text{the set of functions in } L^{p,\lambda,\gamma}(\mathbb{R}^n): \quad \lim_{r\to 0} \mathfrak{M}_0^{p,\lambda,\gamma}(f,r) = 0. \tag{8}$$

The set $V_0L^{p,\lambda,\gamma}(\mathbb{R}^n)$ is a closed subspace of $L^{p,\lambda,\gamma}(\mathbb{R}^n)$.
Weighted complementary Morrey spaces ${}^{c}L^{p,\lambda,\gamma}(\mathbb{R}^n)$ are similarly defined by the norm

$$\|f\|_{{}^{c}L^{p,\lambda,\gamma}(\mathbb{R}^n)} = \sup_{r>0} \left(\mathfrak{M}_\infty^{p,\lambda,\gamma}(f,r)\right)^{\frac{1}{p}}, \quad \mathfrak{M}_\infty^{p,\lambda,\gamma}(f,r) := r^\lambda \int\limits_{|x|>r} (|f(x)||x|^\gamma)^p \, dx$$

and the vanishing complementary weighted Morrey space

$$V_\infty{}^{c}L^{p,\lambda,\gamma}(\mathbb{R}^n): \quad \lim_{r\to\infty} \mathfrak{M}_\infty^{p,\lambda,\gamma}(f,r) = 0.$$

[V. Guliyev. Integral operators on function spaces on homogeneous groups and on domains in $\mathbb{R}^n$. Doctor's degree, Moscow, Steklov Math. Inst., 1994].

In the one-dimensional case $n = 1$ we consider Morrey and complementary Morrey spaces on the semi-axis $\mathbb{R}_+$ instead of $\mathbb{R}$:

$$\mathfrak{M}_0^{p,\lambda,\gamma}(f,r) = \frac{1}{r^\lambda}\int_0^r |x^\gamma f(x)|^p \, dx \quad \text{and} \quad \mathfrak{M}_\infty^{p,\lambda,\gamma}(f,r) = r^\lambda \int_r^\infty |x^\gamma f(x)|^p \, dx.$$

On operators with homogeneous kernels in Morrey spaces

$$Kf(x) = \int_{\mathbb{R}^n} \mathcal{K}(x,y)f(y)dy, \;\; x \in \mathbb{R}^n$$

with the kernel homogeneous of degree $-n$, i.e. $\mathcal{K}(tx, ty) = t^{-n}\mathcal{K}(x,y), t > 0$,

and invariant with respect to rotations in $\mathbb{R}^n$: $\mathcal{K}[\omega(x), \omega(y)] = \mathcal{K}(x,y)$,
where $\omega : x \to \omega(x), \;\; |\omega(x)| = |x|$, is an arbitrary rotation,

The case L^p : [N.K. Karapetiants and S.G. Samko. Multidimensional integral operators with homogeneous kernels, 1999.]

The case $L^{p,\lambda}$: [N. Samko. Integral operators commuting with dilations and rotations in generalized morrey-type spaces, 2020]

A result from the paper [N.Samko, 2020], principal for the proof

We need the following constants to formulate the theorem.

$$\varkappa(p,\lambda,\gamma) := \int_{\mathbb{R}^n} |\mathcal{K}(e_1,y)| \frac{dy}{|y|^{\frac{n-\lambda}{p}+\gamma}}, \;\; n \geq 2, \quad e_1 = (1,0,\dots,0), \tag{9}$$

with one-dimensional modification

$$\varkappa(p,\lambda,\gamma) := \int_0^\infty |\mathcal{K}(1,y)| \frac{dy}{y^{\frac{1-\lambda}{p}+\gamma}} < \infty, \;\; \text{for } n = 1. \tag{10}$$

Theorem 1

Let $1 \leq p < \infty$, $\lambda \geq 0$ *and* $\gamma \in \mathbb{R}$. *Under the condition* $\varkappa(p, \lambda, \gamma) < \infty$, *the operator* K *is bounded in the spaces* $L^{p,\lambda,\gamma}$ *and* $V_0 L^{p,\lambda,\gamma}$ *and*

$$\|Kf\|_{L^{p,\lambda,\gamma}} \leq \varkappa(p, \lambda, \gamma)\|f\|_{L^{p,\lambda,\gamma}}. \tag{11}$$

When $\mathcal{K}(x, y)$ *is non-negative, the condition* $\varkappa(p, \lambda, \gamma) < \infty$ *is also necessary for the boundedness in both the spaces* $L^{p,\lambda,\gamma}$ *and* $V_0 L^{p,\lambda,\gamma}$.

Moreover, $\varkappa(p, \lambda, \gamma)$ *is the sharp constant*

and when $\lambda > 0$, $f(x) = |x|^{\frac{\lambda-n}{p}-\gamma}$ *is the minimizing function in the case of the space* $L^{p,\lambda,\gamma}$.

On isometry between weighted local Morrey and complementary Morrey spaces

Let

$$Q_\ell f(x) = \frac{1}{|x|^\ell} f\left(\frac{x}{|x|^2}\right), \quad x \in \mathbb{R}^n \setminus 0, \ \ell \in \mathbb{R}, \tag{12}$$

so that $Q_\ell^2 = I$.

Lemma 2

Let $1 \leq p < \infty, \ \lambda \geq 0$ *and* $\gamma \in \mathbb{R}$. *Then the following relations hold:*

$$\mathfrak{M}_0^{p,\lambda,\gamma}(Q_\ell f, r) = \mathfrak{M}_\infty^{p,\lambda,\delta}\left(f, \frac{1}{r}\right) \tag{13}$$

and

$$\|Q_\ell f\|_{L^{p,\lambda,\gamma}} = \|f\|_{{}^cL^{p,\lambda,\delta}}, \tag{14}$$

where $\gamma + \delta = \ell - \frac{2n}{p}$.

Proof

The equality (14) for norms follows from (13). For (13) we have

$$\mathfrak{M}_0^{p,\lambda,\gamma}(Q_\ell f, r) = \frac{1}{r^\lambda} \int\limits_{|x|<r} \left| |x|^{\gamma-\ell} f\left(\frac{x}{|x|^2}\right)\right|^p dx = \frac{1}{r^\lambda}\int_{|y|>\frac{1}{r}} ||y|^{\ell-\gamma} f(y)|^p \frac{dy}{|y|^{2n}}$$

via the change of variables $x = \frac{y}{|y|^2}$ with the Jacobian $|y|^{-2n}$, and we arrive at (13).

Main theorems

We want to prove the mapping properties

$$A : \ L^{p,\lambda,\gamma} \hookrightarrow {}^{c}L^{p,\lambda,\delta} \tag{15}$$

$$A : \ {}^{c}L^{p,\lambda,\gamma} \hookrightarrow L^{p,\lambda,\delta}, \tag{16}$$

where $1 \leq p < \infty, \ \lambda \geq 0$ and $\delta, \gamma \in \mathbb{R}$.

Mappings (15), (16) and similar mappings in vanishing Morrey spaces, clearly show how behavior of functions at the origin (at infinity) influences on behavior of the transform at infinity (at the origin, respectively).

Lemma 3

Each of the mapping properties (15) *and* (16) *may hold only in the case when*

$$\delta + \gamma = \frac{p-2}{p}. \tag{17}$$

Proof

Suppose that (15) takes place:

$$\|Af\|_{{}^cL^{p,\lambda,\delta}} \leq C\|f\|_{L^{p,\lambda,\gamma}}.$$

To show that then (17) necessarily holds, we use the dilation operator

$$\Pi_t f(x) = f(tx), \;\; t > 0.$$

Note that

$$\|\Pi_t f\|_{L^{p,\lambda,\gamma}} = \frac{1}{t^{\frac{1-\lambda}{p}+\gamma}}\|f\|_{L^{p,\lambda,\gamma}} \tag{18}$$

and

$$\|\Pi_t f\|_{{}^cL^{p,\lambda,\gamma}} = \frac{1}{t^{\frac{1+\lambda}{p}+\gamma}}\|f\|_{{}^cL^{p,\lambda,\gamma}} \tag{19}$$

for $1 \leq p < \infty, \;\; \lambda \geq 0$ and $\gamma \in \mathbb{R}$.

As is well known, the trick based on the dilation operator effectively works for integral operators with kernels having any kind of homogeneity, and well known for instance for Riesz potential operators in Lebesgue spaces, [E.M.Stain, 1970].

By our assumption we also have that

$$\|A\Pi_t f\|_{{}^cL^{p,\lambda,\delta}} \leq C\|\Pi_t f\|_{L^{p,\lambda,\gamma}}$$

for all $t > 0$. It is easy to see that $A\Pi_t = \frac{1}{t}\Pi_{\frac{1}{t}}A$. Consequently,

$$\|\Pi_{\frac{1}{t}}Af\|_{{}^cL^{p,\lambda,\delta}} \leq Ct\|\Pi_t f\|_{L^{p,\lambda,\gamma}}$$

Applying the formulas (18) and (19) we arrive at

$$\|Af\|_{{}^cL^{p,\lambda,\delta}} \leq Ct^{\frac{p-2}{p}-\delta-\gamma}\|f\|_{L^{p,\lambda,\gamma}}.$$

Hence (17) should hold.
The case of the mapping property (16) is similarly treated.

In view of this lemma, all the exponents δ and γ appearing in the sequel, will be related to each other by the condition

$$\delta + \gamma = \frac{p-2}{p}.$$

Main theorem

Denote

$$\varkappa_0(p, \lambda, \gamma) := \int_0^\infty |k(t)| \frac{dt}{t^{\frac{1-\lambda}{p}+\gamma}}. \tag{20}$$

and

$$\varkappa_\infty(p, \lambda, \gamma) := \int_0^\infty |k(t)| \frac{dt}{t^{\frac{1+\lambda}{p}+\gamma}}. \tag{21}$$

Theorem 4

Let $1 \leq p < \infty,\ \lambda \geq 0$ *and* $\gamma \in \mathbb{R}$.
If $\varkappa_0(p,\lambda,\gamma) < \infty$, *then* $A : L^{p,\lambda,\gamma} \to {}^{c}L^{p,\lambda,\frac{p-2}{p}-\gamma}$ *and*

$$\|Af\|_{{}^{c}L^{p,\lambda,\frac{p-2}{p}-\gamma}} \leq \varkappa_0(p,\lambda,\gamma)\|f\|_{L^{p,\lambda,\gamma}}. \tag{22}$$

If $\varkappa_\infty(p,\lambda,\gamma) < \infty$, *then* $A : {}^{c}L^{p,\lambda,\gamma} \to L^{p,\lambda,\frac{p-2}{p}-\gamma}$ *and*

$$\|Af\|_{L^{p,\lambda,\frac{p-2}{p}-\gamma}} \leq \varkappa_\infty(p,\lambda,\gamma)\|f\|_{{}^{c}L^{p,\lambda,\gamma}}. \tag{23}$$

If $k(x) \geq 0,\ x \in \mathbb{R}_+$, *then the conditions* $\varkappa_0(p,\lambda,\gamma) < \infty$ *and* $\varkappa_\infty(p,\lambda,\gamma) < \infty$ *are also necessary for the boundedness* (22) *and* (23), *respectively, and the constants in* (22) *and* (23) *are sharp.*
And when $\lambda \neq 0$, *the minimizing functions are* $f(x) = x^{\frac{\lambda-1}{p}-\gamma}$ *and* $f(x) = x^{-\frac{\lambda+1}{p}-\gamma}$, *respectively.*

Proof

We will base ourselves on the isometry between Morrey and complementary Morrey spaces via the operator

$$Q_1 f(x) = \frac{1}{x} f\left(\frac{1}{x}\right), \ x > 0.$$

Direct calculation gives

$$Q_1 A = K,$$

where

$$Kf(x) = \int_0^\infty \mathcal{K}(x,t) f(t) dt, \ \mathcal{K}(x,t) = \frac{1}{x}\, k\left(\frac{t}{x}\right).$$

The kernel $\mathcal{K}(x,t)$ is homogeneous of degree -1. We have

$$\|Q_1 Af\|_{L^{p,\lambda,\gamma}} = \|Kf\|_{L^{p,\lambda,\gamma}}.$$

By Lemma 2 on the isometry, we then have

$$\|Af\|_{{}^{c}L^{p,\lambda,\frac{p-2}{p}-\gamma}} = \|Kf\|_{L^{p,\lambda,\gamma}}. \tag{24}$$

By our Theorem 1 on operators with kernel homogeneous of degree $-n$ in Morrey spaces the operator K is bounded in $L^{p,\lambda,\gamma}$, if

$$\int_0^\infty |\mathcal{K}(1,t)| \frac{dt}{t^{\frac{1-\lambda}{p}+\gamma}} = \int_0^\infty |k(t)| \frac{dt}{t^{\frac{1-\lambda}{p}+\gamma}},$$

i.e. $\varkappa_0(p,\lambda,\gamma) < \infty$. Therefore, under this condition from (24) by Theorem 1 we have

$$\|Af\|_{{}^{c}L^{p,\lambda,\frac{p-2}{p}-\gamma}} \leq \varkappa_0(p,\lambda,\gamma)\|f\|_{L^{p,\lambda,\gamma}}$$

with the sharp constant when $k(x) \geq 0,\ x \in \mathbb{R}_+$, which proves (22).

To prove (23) we proceed as follows

$$Af(x) = \int\limits_0^\infty k(xt)f(t)dt = \int\limits_0^\infty k\left(\frac{x}{t}\right) f\left(\frac{1}{t}\right)\frac{dt}{t^2} = \int\limits_0^\infty \mathcal{K}^*(x,t)(Q_1 f)(t)dt =: K^* Q_1 f(x),$$

where

$$\mathcal{K}^*(x,t) = \frac{1}{t}k\left(\frac{x}{t}\right).$$

Hence

$$\|Af\|_{L^{p,\lambda,\delta}} = \|K^* Q_1 f\|_{L^{p,\lambda,\delta}}, \tag{25}$$

where we chose $\delta = \frac{p-2}{p} - \gamma$. By Theorem 1 the operator K^* is bounded in $L^{p,\lambda,\delta}$ if

$$\int_0^\infty |\mathcal{K}^*(1,t)| \frac{dt}{t^{\frac{1-\lambda}{p}+\delta}} = \int_0^\infty \left|k\left(\frac{1}{t}\right)\right| \frac{dt}{t^{1+\frac{1-\lambda}{p}+\delta}} < \infty,$$

which is nothing else but the condition $\varkappa_\infty(p,\lambda,\gamma) < \infty$. Therefore by the boundedness of the operator K^* in $L^{p,\lambda,\delta}$ and isometry provided by Lemma 2, we arrive at (23).

Necessity of the conditions $\varkappa_0(p,\lambda,\gamma) < \infty$ and $\varkappa_\infty(p,\lambda,\gamma) < \infty$ for the corresponding mapping properties immediately follows from the representations $Q_1 = K,\ A = K^* Q_1$, the same isometry and Theorem 1.

The choice of the minimizing function $f(x) = x^{-\frac{\lambda+1}{p}-\gamma}$ for (22) is dictated by Theorem 1 via the identity $Q_1 A = K$.

As regards the minimizing function f for (23), from the identity $A = K^* Q_1$, it is clear that it should be chosen so that

$$Q_1 f = x^{\frac{\lambda-1}{p}-\delta},$$

from which there follows that $f(x) = x^{-\frac{\lambda+1}{p}-\gamma}$, since $\delta + \gamma = \frac{p-2}{p}$.

The case of vanishing Morrey spaces

Theorem 5

Let $1 \leq p < \infty$, $\lambda > 0$ and $\gamma \in \mathbb{R}$. Then the operator A is bounded from $V_0 L^{p,\lambda,\gamma}$ to $V_\infty{}^c L^{p,\lambda,\delta}$ and from $V_\infty{}^c L^{p,\lambda,\gamma}$ to $V_0 L^{p,\lambda,\delta}$, where $\gamma + \delta = \frac{p-2}{p}$, under the conditions $\varkappa_0(p,\lambda,\gamma) < \infty$ and $\varkappa_\infty(p,\lambda,\gamma) < \infty$, respectively, so that there hold the following "regularity properties"

$$\lim_{r\to 0} \frac{1}{r^\lambda} \int_0^r |y^\gamma f(y)|^p dy = 0 \Rightarrow \lim_{r\to\infty} r^\lambda \int_r^\infty |y^{-\gamma} Af(y)|^p \, y^{p-2} dy = 0, \quad \varkappa_0(p,\lambda,\gamma) < \infty \tag{26}$$

$$\lim_{r\to\infty} r^\lambda \int_r^\infty |y^\gamma f(y)|^p dy = 0 \Rightarrow \lim_{r\to 0} \frac{1}{r^\lambda} \int_0^r |y^{-\gamma} Af(y)|^p \, y^{p-2} dy = 0, \quad \varkappa_\infty(p,\lambda,\gamma) < \infty. \tag{27}$$

The statements of the theorem follow from Theorem 1 and the relation (13):

$$\mathfrak{M}_0^{p,\lambda,\gamma}(Q_\ell f, r) = \mathfrak{M}_\infty^{p,\lambda,\delta}\left(f, \frac{1}{r}\right),$$

from isometry.

Applications to concrete integral transforms

The main result is over. It can be directly applied to many concrete Integral Transforms. We can mention, for example the following transforms:

Laplace transform

Bessel transform of Hankel-type

Bessel transform with McDonald function in the kernel

Mittag-Leffler transform

In application to concrete integral transforms with a concrete kernel $k(t)$, according to the above results, our task reduces to

1) verification that the integrals $\int_0^\infty |k(t)| \frac{dt}{t^{\frac{1-\lambda}{p}+\gamma}}$ and $\int_0^\infty |k(t)| \frac{dt}{t^{\frac{1+\lambda}{p}+\gamma}}$ converge;
2) when $k(t) \geq 0$, to explicitly calculate these integrals, if we want to have the sharp constant.

Laplace transform $\mathcal{L}f(x) = \int\limits_0^\infty e^{-xt} f(t)dt$

Corollary 6

Let $1 \le p < \infty$ *and* $\gamma, \delta \in \mathbb{R}$. *The Laplace transform satisfies mapping properties* (15) *and* (16) *if and only if* $\gamma + \delta = \frac{p-2}{p}$ *and* $1 - \lambda < (1-\gamma)p$ *and* $1 + \lambda < (1-\gamma)p$, *respectively, and*

$$\sup_{r>0} r^{\lambda} \int\limits_r^\infty |x^{\delta} \mathcal{L}f(x)|^p dx \le \Gamma\left(1 - \gamma + \frac{\lambda - 1}{p}\right)^p \sup_{r>0} \frac{1}{r^{\lambda}} \int\limits_0^r |x^{\gamma} f(x)|^p dx, \quad \lambda \ge 0, \tag{28}$$

$$\sup_{r>0} \frac{1}{r^{\lambda}} \int\limits_0^r |x^{\delta} \mathcal{L}f(x)|^p dx \le \Gamma\left(1 - \gamma - \frac{\lambda + 1}{p}\right)^p \sup_{r>0} r^{\lambda} \int\limits_r^\infty |x^{\gamma} f(x)|^p dx, \quad \lambda \ge 0, \tag{29}$$

with the best constants in (28) *and* (29),

continue

and also

$$\lim_{r\to 0}\frac{1}{r^\lambda}\int_0^r |x^\gamma f(x)|^p dx = 0 \Longrightarrow \lim_{r\to\infty} r^\lambda \int_r^\infty |x^\delta \mathcal{L}f(x)|^p dx = 0, \quad \lambda > 0 \tag{30}$$

$$\lim_{r\to\infty} r^\lambda \int_r^\infty |x^\gamma f(x)|^p dx = 0 \Longrightarrow \lim_{r\to 0}\frac{1}{r^\lambda}\int_0^r |x^\delta \mathcal{L}f(x)|^p dx = 0, \quad \lambda > 0, \tag{31}$$

where $1-\lambda < (1-\gamma)p$ in (28) and (30), and $1+\lambda < (1-\gamma)p$ in (29) and (31).

Bessel transform of Hankel-type

For the Bessel-type transform

$$B_{\mu,\nu}f(x) = \int_0^\infty (xy)^\mu J_\nu(xy)f(y)dy,\ \mu,\nu \in \mathbb{R},\ \nu > -\frac{1}{2},\ \ J_\nu(x) = \sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu},$$

the sufficient conditions for the mappings $L^{p,\lambda,\gamma} \hookrightarrow {}^{c}L^{p,\lambda,\delta}$ and ${}^{c}L^{p,\lambda,\gamma} \hookrightarrow L^{p,\lambda,\delta}$, with $\gamma + \delta = \frac{p-2}{p}$ are

$$\gamma - \mu - \nu < \frac{1}{p'} + \frac{\lambda}{p} < \gamma - \mu + \frac{1}{2}$$

and

$$\gamma - \mu - \nu < \frac{1}{p'} - \frac{\lambda}{p} < \gamma - \mu + \frac{1}{2},$$

respectively.

$$\mathcal{K}_{\mu,\nu}f(x) = \int_0^\infty (xt)^\mu K_\nu(xt) f(t)\,dt,$$

where

$$K_\nu(x) = \frac{1}{2}\left(\frac{x}{2}\right)^{-\nu} \int_0^\infty t^{\nu-1} e^{-t-\frac{x^2}{4t}}\,dt, \;\; \nu > 0$$

is the Bessel-type function known as the McDonald function.
It is known that

$$\int_0^\infty \frac{K_\nu(x)}{x^\beta}\,dx = 2^{-\beta-1}\Gamma\left(\frac{1-\beta+\nu}{2}\right)\Gamma\left(\frac{1-\beta-\nu}{2}\right), \;\; \beta+\nu<1, \tag{32}$$

From our general theorems, for the transform $\mathcal{K}_{\mu,\nu}$ we obtain the following.

The operator $\mathcal{K}_{\mu,\nu}$ possesses the mapping properties $L^{p,\lambda,\gamma} \hookrightarrow {}^{c}L^{p,\lambda,\delta}$, and similarly for vanishing Morrey spaces, if and only if

$$\gamma + \nu - \mu < \frac{\lambda}{p} + \frac{1}{p'},$$

with the sharp constant

$$\varkappa_0 = 2^{\frac{\lambda-1}{p}+\gamma-\mu-1}\Gamma\left(\frac{\frac{\lambda}{p}+\frac{1}{p'}-\gamma+\mu+\nu}{2}\right)\Gamma\left(\frac{\frac{\lambda}{p}+\frac{1}{p'}-\gamma+\mu-\nu}{2}\right)$$

.

In the formula for the sharp constant we used the relation (32). For the mapping properties ${}^{c}L^{p,\lambda,\gamma} \hookrightarrow L^{p,\lambda,\delta}$, and and similarly for vanishing Morrey spaces, the statement is formulated in the same way, with only change that λ should be replaced by $-\lambda$ in all the conditions and formulas.

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Thank you for your attention!