Sets of uniqueness for inframonogenic functions[1]

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[1] Joint work with Arsenio Moreno García and Tania Moreno García

Dirac operator and monogenic functions

- The Dirac operator $\partial_{\underline{x}}$ for C^1-functions on $\mathbb{R}^m$ is defined by

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- $e_1, e_2, \ldots e_m$ stand for orthonormal basis vectors of $\mathbb{R}^m$, which subjected to the multiplication rules

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- An $\mathbb{R}_{0,m}$-valued function f is called left monogenic (right monogenic) in Ω if $\partial_{\underline{x}} f = 0$ ($f \partial_{\underline{x}} = 0$) in Ω

Harmonic and inframonogenic functions

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H. Malonek, D. Peña-Peña, F. Sommen. *A Cauchy-Kowalevski Theorem for Inframonogenic Functions*. Math. J. Okayama Univ. 53, 167–172, 2011.

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Inframonogenic functions in linear elasticity theory

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- For any harmonic vector field $\vec{h}$, there exists an inframonogenic $\vec{i}$ such that $\vec{h} + \vec{i}$ solves (1), and vice versa.
- The universal (equilibrium) solutions of (1) are exactly those vector fields $\vec{u}$ simultaneously harmonic and inframonogenic.

Inframonogenic functions in linear elasticity theory

- Moreno García, A., Moreno García, T, A. B, R., Bory Reyes, J: *Decomposition of inframonogenic functions with applications in elasticity theory*. Math Meth Appl Sci. 43:1915-1924, 2020.
- Moreno García, A., Moreno García, T, A. B, R., Bory Reyes, J: *Inframonogenic functions and their applications in three dimensional elasticity theory*. Math. Methods Appl. Sci. 41, no.10, 3622-3631, 2018.

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- Inframonogenic functions are biharmonic
- Inframonogenic functions do not satisfy the *mean value property*.
- There is no *maximum principle* for such functions.

Counterexample

As a simple counterexample consider the ellipsoidal domain

$$\Omega = \{\underline{x} \in \mathbb{R}^m : \ (m-1)x_1^2 + x_2^2 + \cdots + x_m^2 < 1\}$$

and the vector-valued polynomial

$$u(\underline{x}) = [(m-1)x_1^2 + x_2^2 + \cdots + x_m^2 - 1]e_1.$$

Obviously it follows that $u|_{\partial\Omega} = 0$ and after some computation we have $\partial_{\underline{x}} u(\underline{x}) \partial_{\underline{x}} = 0$ in Ω

The problem

In 1993 Hayman and Korenblum proved that a set of k distinct spheres contained together with their respective interiors in a domain Ω is a set of uniqueness for polyharmonic functions of order k in Ω ($\Delta^k f = 0$). This means that any polyharmonic function of order k in Ω vanishing on these k distinct spheres is identical to zero.

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Hayman,W. K., Korenblum, B: Representation and uniqueness theorems for polyharmonic functions, J. Anal. Math. 60, 113–133, 1993.

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Being consequence of the maximum principle, it is easy to conclude that only one sphere is a set of uniqueness for harmonic functions in Ω. This rises the question: Is a sphere a set of uniqueness for inframonogenic functions?

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It is shown that in spaces of odd dimension, the above question has a positive answer. In the case of even-dimensional space we provide examples of non-zero inframonogenic functions, which vanish on a sphere.

Auxiliary Results

Let us introduce the operator $\Psi : \mathbb{R}_{0,m} \mapsto \mathbb{R}_{0,m}$ given by

$$\Psi(a) = \sum_{k=1}^{m} e_k a e_k,$$

for $a \in \mathbb{R}_{0,m}$.

The operator Ψ keeps the subspace $\mathbb{R}_{0,m}^{(k)}$ invariant and moreover we have

$$\Psi(Y_k) = (-1)^{k+1}(m - 2k)Y_k, \tag{2}$$

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Proposition 1

Suppose that m is odd. Then Ψ is a bijective $\mathbb{R}$-linear mapping on $\mathbb{R}_{0,m}$. Moreover, its inverse is given by

$$\Psi^{-1}(a) = \sum_{k=0}^{m} \frac{(-1)^{k+1}}{m-2k}[a]_k.$$

Auxiliary Results

Proposition 2

Let be $f : \mathbb{R}^m \mapsto \mathbb{R}_{0,m}$, then

(1) $\partial_{\underline{x}}(f\,\underline{x}) = (\partial_{\underline{x}} f)\underline{x} + \Psi(f),\ (\underline{x}\,f)\partial_{\underline{x}} = \underline{x}(f\partial_{\underline{x}}) + \Psi(f)$

(2) $\partial_{\underline{x}}(\Psi(f)) = -2f\partial_{\underline{x}} - \Psi(\partial_{\underline{x}} f),\ (\Psi(f))\partial_{\underline{x}} = -2\partial_{\underline{x}} f - \Psi(f\partial_{\underline{x}})$

(3) $\partial_{\underline{x}}(\Psi(f))\partial_{\underline{x}} = \Psi(\partial_{\underline{x}} f \partial_{\underline{x}}),\ \partial_{\underline{x}}^2(\Psi(f)) = \Psi(\partial_{\underline{x}}^2 f)$

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(3) $\partial_{\underline{x}}(\Psi(f))\partial_{\underline{x}} = \Psi(\partial_{\underline{x}} f \partial_{\underline{x}}),\ \partial^2_{\underline{x}}(\Psi(f)) = \Psi(\partial^2_{\underline{x}} f)$

Proposition 3

Suppose that m is odd. A function f is inframonogenic (harmonic) in $\Omega \subset \mathbb{R}^m$ if and only if $\Psi(f)$ is inframonogenic (harmonic) in Ω.

Auxiliary Results[2]

Lemma 1

If f is right monogenic in a star-like domain Ω with centre 0, then there exist uniquely defined functions f_1 and f_2, being left and two-sided monogenic, respectively, and such that in Ω:

$$f(\underline{x}) = f_1(\underline{x}) + \underline{x} f_2(\underline{x}). \qquad (3)$$

[2] *Comparing harmonic and inframonogenic functions in Clifford analysis*. To appear in Mediterranean Journal of Mathematics, 2022.

Proof of Lemma 1

Let r be small enough so that the ball $B_r(0)$ together with its boundary is contained in Ω. Since f is right monogenic, then it is also bimonogenic (harmonic) in Ω and by the Green formula we have

$$f(\underline{x}) = \frac{r^2 - |\underline{x}|^2}{r\sigma_m} \int\limits_{\partial B_r(0)} \frac{f(\underline{y})}{|\underline{y} - \underline{x}|^m} d\underline{y}, \ \underline{x} \in B_r(0). \tag{4}$$

Proof of Lemma 1

On the other hand, for $\underline{y} \in \partial B_r(0)$ ($|\underline{y}| = r$) we have

$$\frac{1}{\sigma_m}\frac{r^2 - |\underline{x}|^2}{|\underline{y}-\underline{x}|^m} = \frac{1}{\sigma_m}\left[\frac{-(\underline{y}-\underline{x})\underline{y} - \underline{x}(\underline{y}-\underline{x})}{|\underline{y}-\underline{x}|^m}\right] = E_0(\underline{y}-\underline{x})\underline{y} + \underline{x}E_0(\underline{y}-\underline{x})$$

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Then formula (4) can be rewritten in the form

$$f(\underline{x}) = \frac{1}{r}\int_{\partial B_r(0)} E_0(\underline{y} - \underline{x})\underline{y}f(\underline{y})d\underline{y} + \frac{\underline{x}}{r}\int_{\partial B_r(0)} E_0(\underline{y} - \underline{x})f(\underline{y})d\underline{y}, \qquad (5)$$

Proof of Lemma 1

Now let us consider the functions f_1^*, f_2^* defined in $B_r(0)$ by

$$f_1^*(\underline{x}) = \frac{1}{r} \int_{\partial B_r(0)} E_0(\underline{y} - \underline{x})\underline{y} f(\underline{y}) d\underline{y}, \; f_2^*(\underline{x}) = \frac{1}{r} \int_{\partial B_r(0)} E_0(\underline{y} - \underline{x}) f(\underline{y}) d\underline{y}$$

By (5) it follows that

$$f(\underline{x}) = f_1^*(\underline{x}) + \underline{x} f_2^*(\underline{x}), \; \underline{x} \in B_r(0), \tag{6}$$

where both functions f_1^*, f_2^* are left monogenic. In fact, we can do even better and prove that f_2^* also is right monogenic in $B_r(0)$.

Proof of Lemma 1

From these first steps, the proof is easily completed as follows: the bimonogenic function f can be uniquely represented in Ω as $f = f_1 + \underline{x} f_2$, where f_1, f_2 are left monogenic functions in Ω.

H. Malonek; G. Ren. Almansi-type theorems in Clifford analysis, Math. Methods Appl. Sci. 25 (2002), no. 16-18, 1541–1552.

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This fact together with (6) implies $f_1 = f_1^*$, $f_2 = f_2^*$ in B_r. So the function $f_2 \partial_{\underline{x}}$, being real analytic in Ω identically vanishes in the ball $B_r \subset \Omega$, which leads to $f_2 \partial_{\underline{x}} = 0$ in Ω. This concludes the proof. □

Main Theorem

Theorem

Let f be inframonogenic in $\Omega \subset \mathbb{R}^m$ (m odd) and suppose that f vanishes in some sphere $\partial B_r(\underline{a}) = \{\underline{y} \in \mathbb{R}^m : |\underline{y} - \underline{a}| = r\}$ such that $\overline{B_r(\underline{a})} \subset \Omega$, then f is identically zero in Ω

Proof of the main Theorem

Since translations of inframonogenic functions are inframonogenic, one may without affecting generality, assume that $\underline{a} = 0$ and $\overline{B_r(0)} \subset \Omega$. Let $\epsilon > 0$ be small enough so that $\overline{B_r(0)} \subset B_{r+\epsilon}(0) \subset \Omega$. The inframonogenicity of f in $B_{r+\epsilon}(0)$ implies that $\partial_{\underline{x}} f(\underline{x})$ is right monogenic there.

Proof of the main Theorem

Since $B_{r+\epsilon}(0)$ is a star-like domain with centre 0, it follows from Lemma 1 that $\partial_{\underline{x}} f(\underline{x})$ admits in $B_{r+\epsilon}(0)$ the representation

$$\partial_{\underline{x}} f(\underline{x}) = f_1(\underline{x}) + \underline{x} f_2(\underline{x}),$$

where f_1 and f_2 are left and two-sided monogenic functions, respectively.

Proof of the main Theorem

On the other hand, as it can be easily seen we have

$$\partial_{\underline{x}}\partial_{\underline{x}}[f(\underline{x}) + \frac{1}{2}(r^2 - |\underline{x}|^2)f_2(\underline{x})] = \partial_{\underline{x}}[\underline{x}f_2(\underline{x})] + \partial_{\underline{x}}[-\underline{x}f_2(\underline{x})] = 0.$$

This means that $f(\underline{x}) + \frac{1}{2}(r^2 - |\underline{x}|^2)f_2(\underline{x})$, being harmonic in $B_r(0)$, vanishes on $\partial B_r(0)$. Hence, the maximum principle for harmonic functions implies that

$$f(\underline{x}) = \frac{1}{2}(|\underline{x}|^2 - r^2)f_2(\underline{x}), \ \underline{x} \in B_r(0) \tag{7}$$

and so $\frac{1}{2}(|\underline{x}|^2 - r^2)f_2(\underline{x})$ is inframonogenic in $B_r(0)$.

Proof of the main Theorem

Thus

$$\partial_{\underline{x}}[\frac{1}{2}(|\underline{x}|^2 - r^2)f_2(\underline{x})]\partial_{\underline{x}} = \Psi(f_2(\underline{x})) = 0, \ \underline{x} \in B_r(0), \tag{8}$$

which, together with the bijectivity of Ψ will force f_2 and so f to be identically zero in the whole ball $B_r(0)$. Finally, using the fact that every inframonogenic function is real analytic we obtain $f \equiv 0$ in Ω, concluding the proof. □

The case of even dimension

When the dimension m is even we can provide examples of non-zero inframonogenic functions in $\mathbb{R}^m$ whose restrictions to $\partial B_r(0)$ vanish.

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When the dimension m is even we can provide examples of non-zero inframonogenic functions in $\mathbb{R}^m$ whose restrictions to $\partial B_r(0)$ vanish.

Indeed let $m = 2k$ and choose an arbitrary $\mathbb{R}^{(k)}_{0,m}$-valued left (and so right) monogenic function F_k in $\mathbb{R}^m$. Then the non-zero function $u = (|\underline{x}|^2 - r^2)F_k$, being inframonogenic vanishes on $\partial B_r(0)$.

The case of even dimension

In fact we have

$$\partial_{\underline{x}}[(|\underline{x}|^2 - r^2)F_k] = 2\underline{x}F_k$$

and hence

$$\partial_{\underline{x}}[(|\underline{x}|^2 - r^2)F_k]\partial_{\underline{x}} = 2[\underline{x}F_k]\partial_{\underline{x}} = \Psi(F_k) = 0,$$

where use has been of Proposition 2 and the last equality is a direct consequence of (2). □

Set of uniqueness for infrapolymonogenic functions

Infrapolymonogenic functions $\partial_{\underline{x}}^{2k-1} f \partial_{\underline{x}} = 0$.

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Are k distinct spheres a set of uniqueness for infrapolymonogenic functions?