Inverse scattering for the half line matrix Schrödinger operator

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December 9, 2021

Regional Mathematical Center of Southern Federal University
Seminar on Analysis, Differential Equations and Mathematical Physics

(joint with Ricardo Weder, UNAM, Mexico)

Outline

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Matrix Schrödinger equation on the half line

- $-\psi''(k,x) + V(x)\,\psi(k,x) = k^2\,\psi(k,x), \qquad x \in \mathbb{R}^+$
- $\mathbb{R}^+ := (0, +\infty)$, half line, $x > 0$; prime: x-derivative
- k^2 : spectral parameter
- potential $V(x)$: $n \times n$ matrix-valued, at times n-column vector
- bound-state wavefunction ψ: square-summable n-column vector

Relevant references for the characterization

- Z. S. Agranovich and V. A. Marchenko, *The inverse problem of scattering theory,* Gordon and Breach, New York, 1963.
- T. Aktosun and R. Weder, *Inverse scattering on the half line for the matrix Schrödinger equation,* J. Math. Phys. Anal. Geom. **14** (2018), 237–269.
- T. Aktosun and R. Weder, *Direct and inverse scattering for the matrix Schrödinger equation,* Springer Nature, May 2020.
- M. S. Harmer, *The matrix Schrödinger operator and Schrödinger operator on graphs,* PhD thesis, University of Auckland, New Zeeland, 2004.

Applications

- scattering in quantum mechanics with internal structure (spin)
- quantum wires, quantum circuits
- scattering on quantum graphs
- boundary conditions at vertices of the graph

Selfadjoint boundary condition at $x = 0$

- $-B^\dagger \psi(0) + A^\dagger \psi'(0) = 0$

$$\begin{cases} A \text{ and } B \text{ constant } n \times n \text{ matrices} \\ A^\dagger A + B^\dagger B > 0 \quad \text{and} \quad B^\dagger A = A^\dagger B \end{cases}$$

- dagger †: matrix adjoint (transpose and complex conjugate)
- Dirichlet: with $A = 0$ and $B = -I$, $\quad \psi(0) = 0$

Selfadjoint boundary condition at $x = 0$

- $-B^\dagger\,\psi(0) + A^\dagger\,\psi'(0) = 0$
- boundary condition unaffected by invertible T in $(A, B) \mapsto (AT, BT)$
- $A^\dagger A + B^\dagger B > 0$ is the same as having the rank of $\begin{bmatrix} A \\ B \end{bmatrix}$ to be n
- analogy from the scalar ($n = 1$) case

$$(\cos\theta)\,\psi(0) + (\sin\theta)\,\psi'(0) = 0, \qquad \theta \in (0, \pi]$$

Potential $V(x)$, $n \times n$ matrix-valued

- $V(x)^\dagger = V(x)$, selfadjoint potential
- $V(x) \in L^1_1(\mathbb{R}^+)$: $\displaystyle\int_0^\infty dx\,(1+x)\,|V(x)| < +\infty$

  $|V(x)|$ matrix operator norm
- $\displaystyle\int_0^\infty dx\,|V(x)| < +\infty$ sufficient for results with $k \in \overline{\mathbb{C}^+} \setminus \{0\}$
- physical solution $\Psi(k,x)$ satisfies Dirichlet/non-Dirichlet b.c.

- Agranovich–Marchenko: $\displaystyle\int_0^\infty dx\, x\,|V(x)| < +\infty$

  physical solution $\Psi(k,x)$ satisfies only Dirichlet b.c., not non-Dirichlet b.c.

Faddeev class of input data sets **D**

- **D**: input data set; potential $V(x)$ and the boundary condition
- $-B^\dagger\,\psi(0) + A^\dagger\,\psi'(0) = 0$ with $B^\dagger A = A^\dagger B$ and $A^\dagger A + B^\dagger B > 0$
- $\mathbf{D} := \{V(x), A, B\}$ with uniqueness up to $(A, B) \mapsto (AT, BT)$
- $V(x)^\dagger = V(x)$ and $\displaystyle\int_0^\infty dx\,(1+x)\,|V(x)| < +\infty$

Marchenko class of scattering data sets $\mathbf{S}$

- scattering data set $\mathbf{S} := \{S(k), \{\kappa_j, M_j\}_{j=1}^N\}$
- scattering matrix $S(k)$ for $k \in \mathbb{R}$, $n \times n$ matrix-valued
- κ_j : distinct, positive numbers; N : nonnegative integer
- M_j : $n \times n$ constant matrix, nonnegative, hermitian, rank m_j

  M_j normalization matrix, Marchenko normalization matrix, norming matrix
- Marchenko class: $\mathbf{S}$ satisfies properties $(\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4})$

Marchenko class of scattering data sets **S**

- scattering data set $\mathbf{S} := \{S(k), \{\kappa_j, M_j\}_{j=1}^N\}$
- $\{\kappa_j, M_j\}_{j=1}^N\}$: bound-state data; $-\kappa_j^2$ bound-state energy
- M_j has rank $m_j,$ multiplicity of bound state with κ_j
- N number of bound states without counting multiplicities

$\mathcal{N} := \sum_{j=1}^{N} m_j,$ number of bound states including multiplicities

Faddeev class of input data sets **D**

- **D**: input data set; potential $V(x)$ and the boundary condition
- $-B^\dagger\,\psi(0) + A^\dagger\,\psi'(0) = 0$ with $B^\dagger A = A^\dagger B$ and $A^\dagger A + B^\dagger B > 0$
- $\mathbf{D} := \{V(x), A, B\}$ with uniqueness up to $(A, B) \mapsto (AT, BT)$
- $V(x)^\dagger = V(x)$ and $\displaystyle\int_0^\infty dx\,(1+x)\,|V(x)| < +\infty$

Direct and inverse scattering problems

- Direct problem: $\mathbf{D} \mapsto \mathbf{S}$, Inverse problem: $\mathbf{S} \mapsto \mathbf{D}$
- existence
- uniqueness
- construction, reconstruction
- characterization

Our main results

- 1-to-1 correspondence between Marchenko class and Faddeev class
- construction in $\mathbf{D} \mapsto \mathbf{S}$, construction in $\mathbf{S} \mapsto \mathbf{D}$, $\mathbf{S} \mapsto \mathbf{D} \mapsto \mathbf{S}$
- Dom(Inverse map): the “Marchenko class" of scattering data sets
- Dom(Direct map): the “Faddeev class" of input data sets
- Domain of the inverse map = Range of the direct map
- various equivalent descriptions of the Marchenko class

Trouble with the traditional formulation

- tradition: Dirichlet/non-Dirichlet b.c. is a part of scattering data set
- ours: Boundary condition should be a part of the input data set
- tradition: normalize $S(\pm\infty) = I$
- ours: do not normalize $S(\pm\infty) = I$
- tradition: define $S(k)$ differently with Dirichlet/non-Dirichlet b.c.
- ours: define $S(k)$ the same way with Dirichlet/non-Dirichlet b.c.

Traditional formulation, Ill-posedness of the inverse problem

- $\{S(k) \equiv I, \text{no bound states}, \text{Dirichlet b.c.}\} \mapsto \{V(x) \equiv 0\}$
- $\{S(k) \equiv I, \text{no bound states}, \text{Neumann b.c.}\} \mapsto \{V(x) \equiv 0\}$
- $\{S(k) \equiv I, \text{no bound states}\} \mapsto \begin{cases} \{V(x) \equiv 0, \text{Dirichlet b.c.}\} \\ \{V(x) \equiv 0, \text{Neumann b.c.}\} \end{cases}$
- Inverse problem is ill posed unless b.c. is part of scattering data
- Dirichlet/non-Dirichlet b.c. cannot be mixed

Our formulation, Well-posedness of the inverse problem

- scattering matrix $S(k)$ is properly defined
- b.c. is not a part of scattering data set $\mathbf{S}$
- b.c. is to be recovered from scattering data set $\mathbf{S}$
- Inverse problem is well posed
- Dirichlet/non-Dirichlet b.c. may be mixed

Direct problem: Construction for $\mathbf{D} \mapsto \mathbf{S}$

- $\mathbf{D} := \{V(x), A, B\}$ and $\mathbf{S} := \{S(k), \{\kappa_j, M_j\}_{j=1}^N\}$
- uniquely construct the Jost solution $f(k,x)$ by using $V(x)$ in
$$\begin{cases} -f''(k,x) + V(x)\,f(k,x) = k^2\,f(k,x), \\ f(k,x) = e^{ikx}\,[I + o(1)]\,, \qquad x \to +\infty. \end{cases}$$
- uniquely construct the Jost matrix $J(k)$ by using A, B, $f(k,x)$ via .06in
$J(k) = f(-k^*,0)^\dagger\,B - f'(-k^*,0)^\dagger\,A, \qquad k \in \overline{\mathbb{C}^+}.$
- uniquely construct the scattering matrix $S(k)$ by using $J(k)$ via
$S(k) = -J(-k)\,J(k)^{-1}, \qquad k \in \mathbb{R}.$
- uniquely construct the physical solution $\Psi(k,x)$ via
$\Psi(k,x) = f(-k,x) + f(k,x)\,S(k), \quad k \in \mathbb{R}.$

Direct problem: Construction for $\mathbf{D} \mapsto \mathbf{S}$

- $\mathbf{D} := \{V(x), A, B\}$ and $\mathbf{S} := \{S(k), \{\kappa_j, M_j\}_{j=1}^N\}$
- uniquely determine $\{\kappa_j\}_{j=1}^N$ as zeros $k = i\kappa_j$ of $\det[J(k)]$ in $k \in \mathbb{C}^+$.
- uniquely determine the orthogonal, hermitian $n \times n$ projection matrices $\{P_j\}_{j=1}^N$ projecting $\mathbb{C}^n$ onto $\text{Ker}[J(i\kappa_j)^\dagger]$, $\quad P_j = P_j, \quad P_j^\dagger = P_j.$
- uniquely construct the $n \times n$ positive, hermitian matrices $\{B_j\}_{j=1}^N$ via $B_j := (I - P_j) + P_j \int_0^\infty dx\, f(i\kappa_j, x)^\dagger\, f(i\kappa_j, x)\, P_j.$
- uniquely construct the nonnegative, hermitian normalization matrices $\{M_j\}_{j=1}^N$ via $M_j = B_j^{-1/2}\, P_j.$

Inverse problem: Construction for $\mathbf{S} \mapsto \mathbf{D}$

- $\mathbf{S} := \{S(k), \{\kappa_j, M_j\}_{j=1}^N\}$ and $\mathbf{D} := \{V(x), A, B\}$
- uniquely construct the constant $n \times n$ matrices S_∞ and G_1 by using
$$S(k) = S_\infty + \frac{G_1}{ik} + o\left(\frac{1}{k}\right), \qquad k \to \pm\infty.$$
- uniquely construct the $n \times n$ matrices $F_s(y)$ and $F(y)$ as
$$\begin{cases} F_s(y) := \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} dk\, [S(k) - S_\infty]\, e^{iky}, & y \in \mathbb{R}, \\ F(y) := F_s(y) + \displaystyle\sum_{j=1}^N M_j^2\, e^{-\kappa_j y}, & y \in \mathbb{R}^+. \end{cases}$$
- for each fixed $x \geq 0$, determine $K(x, y)$ as the unique solution in $L^1(x < y < +\infty)$ to the Marchenko integral equation
$$K(x, y) + F(x + y) + \int_x^\infty dz\, K(x, z)\, F(z + y) = 0, \qquad 0 \leq x < y.$$

Inverse problem: Construction for $\mathbf{S} \mapsto \mathbf{D}$

- $\mathbf{S} := \{S(k), \{\kappa_j, M_j\}_{j=1}^N\}$ and $\mathbf{D} := \{V(x), A, B\}$
- uniquely construct the potential $V(x)$ via $V(x) = -2\,\dfrac{d\,K(x,x^+)}{dx}$.
- using S_∞, G_1, $K(0,0)$, obtain the boundary matrices A and B (unique up to a postmultiplication by an invertible matrix T) by solving

$$\begin{cases}(I - S_\infty)\,A = 0,\\ (I + S_\infty)\,B = [G_1 - S_\infty\,K(0,0) - K(0,0)\,S_\infty]\,A.\end{cases}$$

- uniquely construct the Jost solution as $f(k,x) = e^{ikx} + \int_x^\infty dy\,K(x,y)\,e^{iky}$.
- construct the physical solution as $\Psi(k,x) = f(-k,x) + f(k,x)\,S(k)$.
- construct the Jost matrix as $J(k) = f(-k^*,0)^\dagger B - f'(-k^*,0)^\dagger A$.

Verification in $\mathbf{S} \mapsto \mathbf{D}$

- $\mathbf{S} := \{S(k), \{\kappa_j, M_j\}_{j=1}^N\}$ and $\mathbf{D} := \{V(x), A, B\}$
- constructed $V(x)$ is hermitian: $V(x)^\dagger = V(x)$.
- constructed $V(x) \in L_1^1(\mathbb{R}^+)$: $\int_0^\infty dx\,(1+x)\,|V(x)| < +\infty.$
- constructed A and B satisfy $A^\dagger A + B^\dagger B > 0$ and $B^\dagger A = A^\dagger B$.
- constructed physical sol $\Psi(k,x)$ satisfies $-B^\dagger \Psi(k,0) + A^\dagger\,\Psi'(k,0) = 0$.
- constructed bound states $\{\Psi_j(x)\}_{j=1}^N$ satisfy $-B^\dagger \Psi_j(0) + A^\dagger\,\Psi_j'(0) = 0$.

Marchenko class of scattering data sets $\mathbf{S} := \{S(k), \{\kappa_j, M_j\}_{j=1}^N\}$

- Marchenko class: $\mathbf{S}$ satisfies properties $(\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4})$

$$(\mathbf{1}) : \begin{cases} S(-k) = S(k)^\dagger = S(k)^{-1}, & k \in \mathbb{R}, \\ S(k) = S_\infty + \dfrac{G_1}{ik} + o\left(\dfrac{1}{k}\right), & k \to \pm\infty, \\ F_s(y) \in L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}), & F_s(y) := \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} dk\, [S(k) - S_\infty]\, e^{iky}. \end{cases}$$

$$(\mathbf{2}) : F_s'(y) \in L_1^1(\mathbb{R}^+) : \quad \int_0^\infty dy\,(1+y)\,|F_s'(y)| < +\infty.$$

$(\mathbf{3})$: The physical sol $\Psi(k,x)$ satisfies $-B^\dagger \Psi(k,0) + A^\dagger\, \Psi'(k,0) = 0.$

$(\mathbf{4})$: Let $F(y) := F_s(y) + \sum_{j=1}^N M_j^2\, e^{-\kappa_j y}$. Then, $X(y) \equiv 0$ is the only solution in $L^1(\mathbb{R}^+)$ to

$$X(y) + \int_0^\infty dz\, X(z)\, F(z+y) = 0, \qquad y \in \mathbb{R}^+.$$

Characterization of the scattering data in $\mathbf{S} \mapsto \mathbf{D}$

- $\mathbf{S} := \{S(k), \{\kappa_j, M_j\}_{j=1}^N\}$ and $\mathbf{D} := \{V(x), A, B\}$
- 1-to-1 correspondence between Marchenko class and Faddeev class
- $\mathbf{S}$ in Marchenko class, $\mathbf{D}$ in Faddeev class, $\mathbf{S} \mapsto \mathbf{D} \mapsto \mathbf{S}$
- Marchenko class: $\mathbf{S}$ satisfies properties $(\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4})$
- Faddeev class: potential with $V(x)^\dagger = V(x)$ and $\int_0^\infty dx\,(1+x)\,|V(x)| < +\infty$

  and b.c. with boundary matrices with $B^\dagger A = A^\dagger B$ and $A^\dagger A + B^\dagger B > 0$
- equivalent characterization formulations: $\mathbf{S}$ satisfies $(\mathbf{1}^*, \mathbf{2}^*, \mathbf{3}^*, \mathbf{4}^*)$

Many other formulations of characterization the scattering data

- $\mathbf{S}$ in Marchenko class, $\quad \mathbf{D}$ in Faddeev class
- Marchenko class: $\mathbf{S}$ satisfies properties $(\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4})$
- Marchenko class: $\mathbf{S}$ satisfies properties $(\mathbf{1}, \mathbf{2}, \mathbf{III} + \mathbf{V}, \mathbf{4})$
- Marchenko class: $\mathbf{S}$ satisfies properties $(\mathbf{1}, \mathbf{2}, \mathbf{L}, \mathring{\mathbf{4}} + \mathring{\mathbf{5}})$
- Marchenko class: $\mathbf{S}$ satisfies properties $(\mathbf{I} + \mathbf{VI}, \mathbf{2}, \mathbf{A}, \mathbf{4})$
- several equivalent formulations for each of $(\mathbf{3})$, $(\mathbf{4})$, $(\mathbf{III})$, $(\mathbf{V})$

  $(\mathbf{3}_a)$, $(\mathbf{3}_b)$; $\quad (\mathbf{4}_a)$, $(\mathbf{4}_b)$, $(\mathbf{4}_c)$, $(\mathbf{4}_d)$, $(\mathbf{4}_e)$; $\quad (\mathbf{III}_a)$, $(\mathbf{III}_b)$, $(\mathbf{III}_c)$

  $(\mathbf{V}_a)$, $(\mathbf{V}_b)$, $(\mathbf{V}_c)$, $(\mathbf{V}_d)$, $(\mathbf{V}_e)$, $(\mathbf{V}_f)$, $(\mathbf{V}_g)$, $(\mathbf{V}_h)$

Properties (**III**) and (**V**)

(**III**) : $F_s' \in L^1(\mathbb{R}^-) \oplus L^2(\mathbb{R}^-)$, and $X(y) \equiv 0$ is the only solution in $L^2(\mathbb{R}^-)$ to

$$-X(y) + \int_{-\infty}^{0} dz\, X(z)\, F_s(z+y) = 0, \qquad y \in \mathbb{R}^-.$$

(**V**) : There are precisely $\sum_{j=1}^{N} \text{rank}[M_j]$ linearly independent solutions in $L^1(\mathbb{R}^+)$ to

$$X(y) + \int_{0}^{\infty} dz\, X(z)\, F_s(z+y) = 0, \qquad y \in \mathbb{R}^+.$$

Properties (**L**), ($\overset{\circ}{\mathbf{4}}$), ($\overset{\circ}{\mathbf{5}}$)

$$(\mathbf{L}): \begin{cases} S(k) \text{ is continuous in } k \in \mathbb{R}, \\ \arg[\det S(0^+)] - \arg[\det S(+\infty)] = \pi\,[2\mathcal{N} + \mu - n + n_{\mathrm{D}}]\,, \\ \qquad \mu = \text{multiplicity of eigenvalue } +1 \text{ of } S(0), \\ \qquad n_D = \text{multiplicity of eigenvalue } -1 \text{ of } S_\infty, \\ \qquad \mathcal{N} = \sum_{j=1}^{N} \operatorname{rank}[M_j]. \end{cases}$$

($\overset{\circ}{\mathbf{4}}$) : Let $F(y) := F_s(y) + \sum_{j=1}^{N} M_j^2\, e^{-\kappa_j y}$. Then, $X(y) \equiv 0$ is the only solution in $L^2(\mathbb{R}^+)$ to

$$X(y) + \int_0^\infty dz\, X(z)\, F(z+y) = 0, \qquad y \in \mathbb{R}^+.$$

($\overset{\circ}{\mathbf{5}}$) : $F_s' \in L^1(\mathbb{R}^-) \oplus L^2(\mathbb{R}^-)$.

Properties (**I**), (**VI**), (**A**)

$$
(\mathbf{I}): \begin{cases} S(-k) = S(k)^\dagger = S(k)^{-1}, \qquad k \in \mathbb{R}, \\ S(k) = S_\infty + O\left(\dfrac{1}{k}\right), \qquad k \to \pm\infty, \\ F_s(y) \in L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}), \quad F_s(y) := \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} dk\, [S(k) - S_\infty]\, e^{iky}. \end{cases}
$$

(**VI**) : $S(k)$ is continuous in $k \in \mathbb{R}$.

$$
(\mathbf{A}): \begin{cases} \text{There exists at least one solution } h(k) \in \mathbf{H}^2(\mathbb{C}^+) \text{ to} \\ \qquad h(-k) + S(k)\, h(k) = g(k), \qquad k \in \mathbb{R}, \\ \text{for every } g(k) \text{ in a dense subspace of } L^2(\mathbb{R}) \text{ satisfying} \\ \qquad g(-k) = S(k)\, g(k), \qquad k \in \mathbb{R}. \end{cases}
$$

Example 1: The scattering data in the Marchenko class

$n = 1, \; S(k) = -\dfrac{k+i}{k-i},$ no bound states.

This scattering data set belongs to the Marchenko class.

$$S(0) = 1, \quad S_\infty = -1, \quad G_1 = 2, \quad \mu = 1, \quad n_{\mathrm{D}} = 1, \quad \mathcal{N} = 0,$$

$$F_s(y) = \begin{cases} 2e^{-y}, & y \in \mathbb{R}^+, \\ 0, & y \in \mathbb{R}^-, \end{cases}$$

$$K(0,0) = -1, \quad f(k,x) = e^{ikx}\left[1 - \frac{i}{k+i}\,\frac{e^{-x}}{\cosh x}\right],$$

$$V(x) = -2\,\mathrm{sech}^2\, x, \quad \Psi(k,x) = -\frac{2ik}{k-i}\,\sin kx - \frac{1}{k-i}\,(\cos kx)(\tanh x),$$

$A = 0, \quad J(k) = \dfrac{k}{k+i}\,B, \quad \Psi(k,0) = 0, \quad \Psi'(k,0) = -2i(k+i)$, where B is an arbitrary nonzero constant.

Example 2: The scattering data not in the Marchenko class

$n = 1, \;\; S(k) = \dfrac{k}{k+i},$ no bound states.

This scattering data set does not belong to the Marchenko class even though $(\mathbf{2}, \mathbf{III} + \mathbf{V}, \mathbf{4})$ hold. Only $(\mathbf{1})$ does not hold: $S(k)^\dagger \neq S(k)^{-1}$.

$(\mathbf{L})$ does not hold because $\mathcal{N} = -1/2$, not a nonnegative integer.

$\mathbf{S} \mapsto \mathbf{D} \not\mapsto \mathbf{S}, \qquad \mathbf{D}$ yields $S(k) = \dfrac{k - i/2}{k + i/2}.$

Example 3: The scattering data not in the Marchenko class

$n = 1, \;\; S(k) = i\dfrac{k-i}{k+i}, \;\;$ no bound states.

This scattering data set does not belong to the Marchenko class even though $(\mathbf{2}, \mathbf{III} + \mathbf{V}, \mathbf{4})$ hold. Only $(\mathbf{1})$ does not hold: $S(-k) \neq S(k)^\dagger$.

$A = B = 0, \;\;$ not satisfying $\;A^\dagger A + B^\dagger B > 0.$

Example 4: The scattering data not in the Marchenko class

$n = 1, \; S(k) = \dfrac{k+i}{k-i}$, no bound states.

This scattering data set does not belong to the Marchenko class even though $(\mathbf{1}, \mathbf{2}, \mathbf{III})$ hold. Only $(\mathbf{4}, \mathbf{V})$ do not hold.

$$V(x) = \frac{8e^{2x}}{(e^{2x}-1)^2}, \qquad V \notin L_1^1(\mathbb{R}^+),$$

$$V(x) = \frac{2}{x^2} - \frac{2}{3} + \frac{2x^2}{15} + O(x^4), \qquad x \to 0,$$

$K(0,0) = -\infty$, and hence no A and B satisfying $A^\dagger A + B^\dagger B > 0$.

Example 5: The scattering data not in the Marchenko class

$$n = 1,\ S(k) = \left(\frac{k-i}{k+i}\right)^2, \text{ no bound states.}$$

This scattering data set does not belong to the Marchenko class even though (**1**, **2**, **4**, **V**) hold. Only (**III**) does not hold.

$K(0,0) = 0, \quad V(x) \equiv 0, \quad B = 2A$, with A as any nonzero constant.

(**3**) fails: $-B^\dagger\, \Psi(k,0) + A^\dagger\, \Psi'(k,0) \neq 0,$

$$\Psi(k,0) = \frac{2k^2 - 2}{(k+i)^2}, \quad \Psi'(k,0) = \frac{4k^2}{(k+i)^2}.$$

(**L**) fails because $\mathcal{N} = -1$, not a nonnegative integer.

Example 6: The scattering data in the Marchenko class

$$n=2,\ \ S(k)=\frac{1}{(k-i)\left(k-\frac{i}{3}\right)}\begin{bmatrix} k(k+i) & \frac{i}{3}(k+i) \\ \frac{i}{3}(k+i) & k(k+i)\end{bmatrix},\ \text{one bound state at } k=i$$

of multiplicity two with $M_1=\begin{bmatrix} 1+\frac{1}{\sqrt{2}} & 1-\frac{1}{\sqrt{2}} \\ 1-\frac{1}{\sqrt{2}} & 1+\frac{1}{\sqrt{2}}\end{bmatrix}.$

This scattering data set belongs to the Marchenko class

$V(x)=-\dfrac{8e^{2x/3}}{9(2+e^{2x/3})^2}\begin{bmatrix} 1 & 1 \\ 1 & 1\end{bmatrix},\quad B=\dfrac{1}{18}\begin{bmatrix} -17 & 1 \\ 1 & -17\end{bmatrix} A$, with A being any invertible 2×2 matrix.