Free boundary problems and Riemann-Hilbert problems on Riemann surfaces

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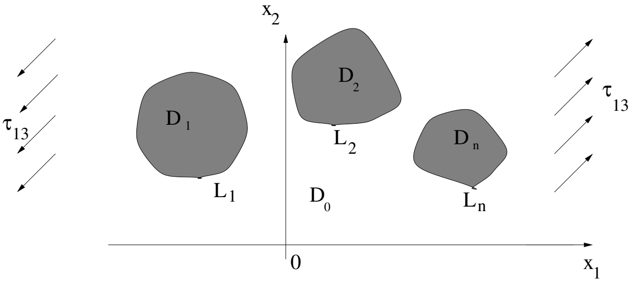
Outline

- Model 1: an inverse antiplane problem in an $(n+1)$-connected domain
- Vector Riemann-Hilbert problem on a genus-n Riemann surface of a family of conformal mapping
- Scalar Riemann-Hilbert problems on the surface
- Numerical results in the elliptic case
- Model 2: Vortex patch in a wedge
- Conformal mapping and a Riemann-Hilbert problem on a torus
- Discussion of numerical results

Setting of Model 1: to appear in Proc Roy Soc A



Let n elastic inclusions $D_1, \ldots D_n$ be embedded into a semi-infinite plane $x_2 > 0$. The shear moduli of D_0 and D_j are μ_0 and μ_j. The inclusions are in ideal contact with D_0. The body is subjected to antiplane shear $\tau_{13} = \tau_1^\infty$ as $x_1 \to \pm\infty$. The boundary of the body $L_0 = \{|x_1| < \infty, x_2 = 0\}$ is free of traction, $\tau_{23} = 0$, and as $x_2 \to \infty$, $\tau_{23} = 0$. Recover the shapes of uniformly stressed inclusions D_j such that in D_j, $\tau_{13} = \tau_1^j$ and $\tau_{23} = \tau_2^j$.

Background: Inverse problems in the whole plane

- *Cherepanov GP. 1974 Inverse problems of the plane theory of elasticity*: Schwarz problem for two symmetric elastic inclusions.
- *Kang H, Kim E, Milton GW. 2008 Inclusion pairs satisfying Eshelby's uniformity property*: Weierstrass zeta function and the Schwarz-Christoffel formula and two antiplane inclusions.
- *Wang X, Yang P, Schiavone P. 2020 Uniform stresses inside a non-elliptical inhomogeneity and a nearby half-plane with locally wavy interface*: Laurent series representation of a conformal map, two antiplane inclusions.
- *Antipov YA. 2019 Method of automorphic functions for an inverse problem of antiplane elasticity.*
- *Antipov YA. 2020 Method of Riemann surfaces for an inverse antiplane problem in an n-connected domain.*

Complex potential f

Let $w_j(x_1, x_2)$, $(x_1, x_2) \in D_j$ $(j = 0, 1, \ldots, n)$, be the x_3-component of the displacement vector. On the interfaces L_j, the boundary conditions of ideal contact read

$$w_0 = w_j, \quad \mu_0 \frac{\partial w_0}{\partial \nu} = \mu_j \frac{\partial w_j}{\partial \nu}, \quad x \in L_j, \quad j = 1, \ldots, n,$$

L_j are the inclusions boundary, and $\frac{\partial}{\partial \nu}$ is the normal derivative. Denote next a harmonic conjugate $w_0^*(x_1, x_2)$ of the function w_0.
Case (i): $\tau_1^j = \tau_1$, $\tau_2^j = \tau_2$,
Introduce a function $f(z)$

$$f(z) = w_0(x_1, x_2) + i w_0^*(x_1, x_2) - \frac{\bar{\tau} z}{\mu_0}, \quad z = x_1 + i x_2 \in D_0,$$

analytic in the domain D_0. Here, $\bar{\tau} = \tau_1 - i\tau_2$. Then

$$f(z) = \frac{1}{\lambda_j} \operatorname{Re} \frac{\bar{\tau} z}{\mu_0} + a_j + i b_j, \quad z \in L_j, \quad j = 1, 2, \ldots, n,$$

$\lambda_j = \kappa_j/(1 - \kappa_j)$, $\kappa_j = \mu_j/\mu_0$, b_j are arbitrary real constants.

Conformal mapping. Interfacial conditions

Let $z = \omega(\zeta)$ be a conformal map $\mathcal{D} \to D_0$ from a parametric ζ-plane cut along segments $l_0 = [m, \infty)$ and $l_j = [k_{2j-1}, k_{2j}], j = 1, 2, \ldots, n,$ $k_1 = 0, k_2 = 1.$

$$k_{2n-1} < k_{2n} < k_{2n-3} < k_{2n-2} < \ldots < k_3 < k_4 < k_1 < k_2 < m.$$

The function $\omega(\zeta)$ maps the two-sided finite segments l_j into the contours L_j $(j = 1, \ldots, n)$ and the two-sided semi-infinite contour l_0 into the x_1-axis of the physical plane.
Introduce a new function $F(\zeta) = f(\omega(\zeta))$ analytic in the domain $\mathcal{D}$.
Then the complex boundary conditions become

$$\operatorname{Im} F(\zeta) = b_j, \quad \operatorname{Re} \frac{\bar{\tau}\omega(\zeta)}{\mu_0} = \lambda_j[\operatorname{Re} F(\zeta) - a_j], \quad \zeta \in l_j. \quad j = 1, \ldots, n.$$

Conditions on the preimage of the half-plane boundary

Since $x_2 = 0$ on the contour L_0,

$$\operatorname{Im}\omega(\zeta) = 0, \quad \zeta \in l_0.$$

Use the condition $\partial w/\partial x_2 \to 0$ as $x_2 \to 0^+$, $-\infty < x_1 < \infty$, and the Cauchy-Riemann conditions. Then $w^* = b_0$ on the line L_0, where b_0 is an arbitrary real constant. Therefore

$$\operatorname{Im} f(z) = b_0 - \operatorname{Im}\frac{\bar{\tau} z}{\mu_0}, \quad z \in L_0.$$

The second condition on the slit l_0 reads

$$\operatorname{Im} F(\zeta) = b_0 - \operatorname{Im}\frac{\bar{\tau}\omega(\zeta)}{\mu_0}, \quad \zeta \in l_0.$$

Conditions at infinity:

$$F(\zeta) \sim \frac{\tau_1^\infty - \bar{\tau}}{\mu_0}\omega(\zeta), \quad \omega(\zeta) \sim c_\pm \xi^{1/2}, \quad \zeta = \xi \pm i0, \quad \xi \to \infty.$$

Riemann surface

Let $\mathcal{R}$ be a genus-n hyperelliptic surface of the algebraic function

$$u^2 = p(\zeta), \quad p(\zeta) = \zeta(1-\zeta)(\zeta - m)\prod_{j=3}^{2n}(\zeta - k_j).$$

Fix a single branch of the function $p^{1/2}(\zeta)$ in the ζ-plane cut along the contours l_j $(j = 0, 1, \dots, n)$ by the condition $p^{1/2}(\xi + i0) = i\sqrt{|p(\xi)|}$, $\xi > m$. On this surface we introduce two functions

$$\Phi_1(\zeta, u) = \begin{cases} F(\zeta), & (\zeta, u) \in \mathcal{D}^+, \\ \overline{F(\bar{\zeta})}, & (\zeta, u) \in \mathcal{D}^-, \end{cases} \quad \Phi_2(\zeta, u) = \begin{cases} i\mu_0^{-1}\bar{\tau}\omega(\zeta), & on\ \mathcal{D}^+, \\ -i\mu_0^{-1}\tau\overline{\omega(\bar{\zeta})}, & on\ \mathcal{D}^-. \end{cases}$$

These functions are symmetric with the respect to the contour $\mathcal{L}$,

$$\Phi_j(\zeta, u) = \overline{\Phi_j(\zeta_*, u_*)}, \quad (\xi, v) \in \mathcal{L},$$

$(\zeta_*, u_*) = (\bar{\zeta}, -u(\bar{\zeta}))$.

Vector Riemann-Hilbert problem on the surface $\mathcal{R}$

$$\Phi^+(\xi, v) = G(\xi, v)\Phi^-(\xi, v) + g(\xi, v), \quad (\xi, v) \in \mathcal{L} \subset \mathcal{R},$$

where $G(\xi, v)$ is a piece-wise constant matrix

$$G(\xi, v) = \begin{pmatrix} 1 & 0 \\ 2i\lambda_j & 1 \end{pmatrix}, (\xi, v) \in l_j, \; G(\xi, v) = \begin{pmatrix} 1 & i(1 - \bar{\tau}/\tau) \\ 0 & -\bar{\tau}/\tau \end{pmatrix} \; on \; l_0,$$

and $g(\xi, v)$ is a piece-wise constant vector

$$g(\xi, v) = \begin{pmatrix} 2ib_j \\ -2i\lambda_j(a_j - ib_j) \end{pmatrix}, (\xi, v) \in l_j, \; g(\xi, v) = \begin{pmatrix} 2ib_0 \\ 0 \end{pmatrix} \; on \; l_0.$$

The vector $\Phi(\zeta, u)$ is symmetric with respect to the contour $\mathcal{L}$, $\Phi(\zeta, u) = \overline{\Phi(\zeta_*, u_*)}$, and its components satisfy the conditions at infinity

$$\Phi_1^+(\zeta, u) \sim \frac{\tau_1^\infty - \bar{\tau}}{i\bar{\tau}}\Phi_2^+(\zeta, u), \quad \Phi_2(\zeta, u) = O(\zeta^{1/2}), \quad \zeta \to \infty.$$

Two scalar Riemann-Hilbert problems on the surface $\mathcal{R}$

Let $\tau_2^j = 0, j = 1, \ldots, n$. Then the vector RH problem is decoupled

$$\Phi_1^+(\xi, v) - \Phi_1^-(\xi, v) = 2ib_j, \quad (\xi, v) \in l_j, \quad j = 0, 1, \ldots, n,$$

$$\Phi_2^+(\xi, v) + \Phi_2^-(\xi, v) = 0, \quad (\xi, v) \in l_0,$$

$$\Phi_2^+(\xi, v) - \Phi_2^-(\xi, v) = 2i\lambda_j[\operatorname{Re}\Phi_1^+(\xi, v) - a_j], \quad (\xi, v) \in l_j, \quad j = 1, \ldots, n.$$

Case (ii): $\tau_1^j/\mu_j = \nu_1$, $\tau_2^j/\mu_j = \nu_2$, $j = 1, 2, \ldots, n$, Then the problem reduces to a similar vector RH problem on the surface $\mathcal{R}$. If $\tau_2^j = 0$, then it is decoupled

$$\Phi_1^+(\xi, v) - \Phi_1^-(\xi, v) = 2ia_j, \quad (\xi, v) \in l_j, \quad j = 1, \ldots, n$$

$$\Phi_1^+(\xi, v) + \Phi_1^-(\xi, v) = -2b_0, \quad (\xi, v) \in l_0,$$

and

$$\Phi_2^+(\xi, v) - \Phi_2^-(\xi, v) = -\frac{2i}{\kappa_j - 1}[\operatorname{Re}\Phi_1^+(\xi, v) + b_j], \ (\xi, v) \in l_j, \ j = 1, \ldots, n,$$

$$\Phi_2^+(\xi, v) - \Phi_2^-(\xi, v) = 0, \quad (\xi, v) \in l_0.$$

Analogue of the Cauchy kernel on a hyperelliptic surface

The Behnke-Stein kernel $Wd\xi = \frac{1}{2}\frac{u+v)d\xi}{v(\xi-\zeta)}$ does no work. We propose

$$V(\xi, v; \zeta, u)d\xi = \frac{1}{2}\left(\frac{\zeta-\xi_0}{\xi-\xi_0} + \frac{u}{v}\prod_{j=1}^{n}\frac{\xi-\xi_j}{\zeta-\xi_j}\right)\frac{d\xi}{\xi-\zeta}.$$

(a) With respect to (ξ, v), $Vd\xi$ is an abelian differential of the 3rd kind. It has simple poles at (ζ, u) and $(\xi_0, \pm u(\xi_0))$. As $\xi \to \infty$, $V = O(\xi^{-3/2})$.
(b) With respect to (ζ, u), $Vd\xi$ is a meromorphic function. It has simple poles at (ξ, v) and $(\xi_j, \pm u(\xi_j))$ The residues of the kernel $Vd\xi$

$$\operatorname*{res}_{(\zeta,u)=(\xi_j,\pm u(\xi_j))} V(\xi, v; \zeta, u)d\xi = \pm\frac{u(\xi_j)}{2}\varphi_j(\xi)d\xi$$

generate a basis of abelian differentials of the first kind

$$\varphi_j(\xi)d\xi = \left(\prod_{s=1,s\neq j}^{n}\frac{\xi-\xi_s}{\xi_j-\xi_s}\right)\frac{d\xi}{v}, \quad j = 1, 2, \ldots, n.$$

$$\Phi_1(\zeta,u)=N_0+iu(\zeta)\sum_{j=1}^{n}\frac{N_j}{\zeta-\xi_j}+\frac{1}{\pi}\sum_{j=0}^{n}b_j\int_{l_j}V(\xi,v;\zeta,u)d\xi.$$

The points $(\xi_j,\pm u(\xi_j))$ are removable singularities of the function $\Phi_1(\zeta,u)$ if and only if

$$N_j=-\frac{1}{2\pi i}\sum_{s=0}^{n}b_s\int_{l_s}\varphi_j(\xi)d\xi,\quad j=1,2,\dots,n.$$

The solution is growing at infinity as $\zeta^{n-1/2}$. To satisfy the conditions at infinity for $n\geq 2$, we require

$$\lim_{\zeta\to\infty}\zeta^{-1/2-s}u(\zeta)\sum_{j=1}^{n}\frac{N_j}{\zeta-\xi_j}=0,\quad s=1,2,\dots,n-1.$$

In the elliptic case, $n=1$, $b_1=b_0-\frac{\pi N_1}{2k\mathbb{K}}$.

The second Riemann-Hilbert problem

$$\Phi_2^+(\xi, v) = G_2(\xi, v)\Phi_2^-(\xi, v) + g_2(\xi, v) \quad (\xi, v) \in \mathcal{L},$$

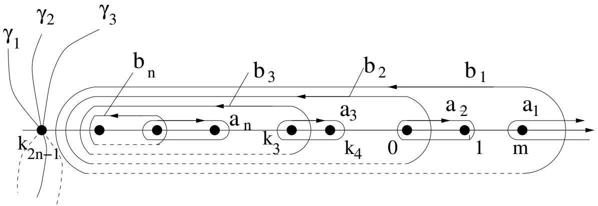
where

$$G_2(\xi, v) = \left\{ \begin{array}{cc} -1, & on\ l_0, \\ 1, & on\ \mathcal{L}', \end{array} \right. \quad g_2(\xi, v) = \left\{ \begin{array}{cc} 0, & on\ l_0, \\ 2i\lambda_j[\mathrm{Re}\,\Phi_1^+(\zeta, v) - a_j], & on\ l_j. \end{array} \right.$$

$\mathcal{L}' = \cup_{j=1}^n l_j$. The function $\Phi_2(\zeta, u)$ has to satisfy the symmetry condition $\Phi_2(\zeta, u) = \overline{\Phi_2(\zeta_*, u_*)}$, and the condition at infinity

$$\Phi_1^+(\zeta, u) \sim \frac{\tau_1^\infty - \bar{\tau}}{i\bar{\tau}}\Phi_2^+(\zeta, u), \quad \Phi_2(\zeta, u) = O(\zeta^{1/2}), \quad \zeta \to \infty.$$

Factorization problem



$$G_2(\xi,v) = X^+(\xi,v)[X^-(\xi,v)]^{-1}, \ (\xi,v)\in\mathcal{L}, \ X(\zeta,u) = \overline{X(\zeta_*,u_*)}$$

The solution of the factorization problem is

$$X(\zeta,u) = \chi(\zeta,u)\overline{\chi(\zeta_*,u_*)}\exp\left\{\frac{1}{2\pi i}\int_{l_0}\log(-1)dV(\xi,v;\zeta,u))\right\}$$

$$\chi(\zeta,u) = \exp\left\{-\sum_{j=1}^{n}\left(\int_{\gamma_j} + n_j\int_{\mathbf{a}_j} + m_j\int_{\mathbf{b}_j}\right)V(\xi,v;\zeta,u)\right\}.$$

Jacobi inversion problem

The function $X(\zeta, u)$ has essential singularities at the simple poles of the kernel $V(\xi, v; \zeta, u)$ They are removable singular points iff

$$\frac{1}{2}\int_{l_0}\varphi_s(\xi)d\xi - \sum_{j=1}^{n}\left[\int_{\gamma_j}\varphi_s(\xi)d\xi - \int_{\gamma_j}\varphi_s(\bar{\xi})d\bar{\xi} + 2n_j\int_{\mathbf{a}_j}\varphi_s(\xi)d\xi\right] = 0,$$

$s = 1, 2, \ldots, n$. This is the genus-n Jacobi inversion problem:
Find n points $q_j = (\zeta_j, u_j'')$ on the surface $\mathcal{R}$ and $2n$ integers n_j and m_j $(j = 1, 2, \ldots, n)$ such that

$$\sum_{j=1}^{n}\left(\int_{k_{2n-1}}^{q_j}\varphi_s(\xi)d\xi + n_jA_{sj} + m_jB_{js}\right) = h_s, \quad s = 1, 2, \ldots, n,$$

$$A_{js} = \int_{\mathbf{a}_j}\varphi_s(\xi)d\xi, \quad B_{js} = \int_{\mathbf{b}_j}\varphi_s(\xi)d\xi,$$

$$h_s = \frac{1}{4}A_{1s} + \sum_{j=1}^{n}\int_{k_{2n-1}}^{\eta_j}\varphi_s(\xi)d\xi.$$

$$A_{11}=-2i\int_m^\infty\frac{d\xi}{\sqrt{|p(\xi)|}}=-4ik\mathbb{K},\quad B_{11}=2\int_1^m\frac{d\xi}{\sqrt{|p(\xi)|}}=4k\mathbb{K}',$$

$$\mathbb{K}'=\mathbb{K}(\sqrt{1-k^2}),\quad h_1=-ik\mathbb{K}+\int_0^{\eta_1}\frac{d\xi}{p^{1/2}(\xi)}.$$

The solution of the Jacobi problem is $\zeta_1=\operatorname{sn}^2\frac{ih_1}{2k}$. If the numbers

$$n_1=-\frac{\operatorname{Im}\mathcal{I}_-}{4k\mathbb{K}},\quad m_1=\frac{\operatorname{Re}\mathcal{I}_-}{4k\mathbb{K}'}.$$

are integers, then the point $q_1=(\zeta_1,\sqrt{p(\zeta_1)})\in\mathcal{D}^+$. Otherwise

$$n_1=-\frac{\operatorname{Im}\mathcal{I}_+}{4k\mathbb{K}},\quad m_1=\frac{\operatorname{Re}\mathcal{I}_+}{4k\mathbb{K}'}$$

are integers and $q_1=(\zeta_1,-\sqrt{p(\zeta_1)})\in\mathcal{D}^-$. Here,

$$\mathcal{I}_\pm=\int_0^{\eta_1}\frac{d\xi}{p^{1/2}(\xi)}\pm\int_0^{\zeta_1}\frac{d\xi}{p^{1/2}(\xi)}-ik\mathbb{K}.$$

Function $\Psi(\zeta, u)$

Introduce a singular integral

$$\Psi(\zeta,u)=\frac{1}{2\pi i}\int_{\mathcal{L}'}\frac{g_2(\xi,v)}{X^+(\xi,v)}dV(\xi,v;\zeta,u),\quad (\zeta,u)\in\mathcal{R}.$$

$\mathcal{L}'=\cup_{j=1}^n l_j$. In the elliptic case ($\xi_0=\xi_1$)

$$\Psi(\zeta,u)=\frac{1}{4\pi i}\int_0^1\left[\frac{g_2(\xi^+,v^+)}{X^+(\xi^+,v^+)}\left(\frac{\zeta-\xi_0}{\xi-\xi_0}+\frac{u}{v^+}\frac{\xi-\xi_0}{\zeta-\xi_0}\right)\right.$$
$$\left.-\frac{g_2(\xi^-,v^-)}{X^+(\xi^-,v^-)}\left(\frac{\zeta-\xi_0}{\xi-\xi_0}+\frac{u}{v^-}\frac{\xi-\xi_0}{\zeta-\xi_0}\right)\right]\frac{d\xi}{\xi-\zeta},$$

$$g_2(\xi^\pm,v^\pm)=2i\left[N_0^*\pm\frac{\sqrt{|p(\xi)|}}{\xi-\xi_0}g_0(\xi)\right],$$

$$g_0(\xi)=2N_1-\frac{b_0}{\pi}\int_m^\infty\frac{(\tau-\xi_0)d\tau}{\sqrt{|p(\tau)|}(\tau-\xi)}+\frac{b_1}{\pi}\int_0^1\frac{(\tau-\xi_0)d\tau}{\sqrt{|p(\tau)|}(\tau-\xi)}.$$

Solution of the second RH problem

After factorization of G_2 the 2nd RH problem becomes

$$\frac{\Phi_2^+(\xi,v)}{X^+(\xi,v)} = \frac{\Phi_2^-(\xi,v)}{X^-(\xi,v)} + \frac{g_2(\xi,v)}{X^+(\xi,v)}, \quad (\xi,v) \in \mathcal{L}.$$

Its solution is

$$\Phi_2(\zeta,u) = X(\zeta,u)[\Psi(\zeta,u) + \Omega(\zeta,u)], \quad (\zeta,u) \in \mathcal{R}.$$

The general form of the rational function $\Omega(\zeta,u)$ is

$$\Omega(\zeta,u) = M_0 + \sum_{j=1}^{n} \left[(M_{1j} + iM_{2j}) \frac{u(\zeta) + u(\eta_j)}{\zeta - \eta_j} \right.$$

$$\left. -(M_{1j} - iM_{2j}) \frac{u(\zeta) - u(\bar{\eta}_j)}{\zeta - \bar{\eta}_j} + \frac{iM_{3j}u(\zeta)}{\zeta - \xi_j} \right],$$

M_0, M_{1j}, M_{2j}, and M_{3j} ($j = 1, 2, \ldots, n$) are arbitrary real constants.

Additional conditions

At the points $(\xi_j, \pm u(\xi_j))$, the poles of the kernel $V(\xi, v; \zeta, u)d\xi$,

$$\operatorname*{res}_{\zeta=\xi_j}[\Psi(\zeta, u) + \Omega(\zeta, u)] = 0, \quad j = 1, 2, \ldots, n.$$

These conditions determine M_{3j}

$$M_{3j} = \frac{1}{4\pi}\int_{l_j} \frac{g_2(\xi, v)\varphi_j(\xi)d\xi}{X^+(\xi, v)}.$$

Since $u(\zeta) = O(\zeta^{n+1/2})$, $\zeta \to \infty$, we have $n-1$ conditions at infinity:

$$\lim_{\zeta\to\infty} \frac{u}{\zeta^s}\sum_{j=1}^{n}\left(\frac{M_{1j}+iM_{2j}}{\zeta-\eta_j} - \frac{M_{1j}-iM_{2j}}{\zeta-\bar{\eta}_j} + \frac{iM_{3j}}{\zeta-\xi_j}\right) = 0, \; s = 1, \ldots, n-1,$$

and the nth condition comes from the problem setting

$$\Phi_1^+(\zeta, u) - \frac{\tau_1^\infty - \tau_1}{i\tau_1}\Phi_2^+(\zeta, u) \sim 0, \; \zeta \to \infty.$$

Constants M_0 and M_{1j} in the elliptic case

The limit at infinity of the function $X(\zeta, u)$ is

$$\lim_{\zeta\to\infty} X(\zeta, u) = iX_\infty, \quad X_\infty = \left|\frac{\zeta_1 - \xi_0}{\eta_1 - \xi_0}\right|,$$

Condition at infinity gives

$$M_{21} = -\frac{M_{31}}{2} + \frac{\tau_1 N_1}{2(\tau_1^\infty - \tau_1)X_\infty}.$$

Residue at the pole of the kernel $Vd\xi$ determines

$$M_{31} = -\frac{1}{4\pi i}\int_0^1 \left[\frac{g_2(\xi^+, v^+)}{X^+(\xi^+, v^+)} + \frac{g_2(\xi^-, v^-)}{X^+(\xi^-, v^-)}\right]\frac{d\xi}{\sqrt{|p(\xi)|}}.$$

The complex condition eliminating the pole of X at (ζ_1, u'') gives M_{11} and M_0

$$M_{11} = \frac{(P^+ + P^-)M_{21} + P_0 M_{31} + Q}{Q^- - Q^+},$$

$$M_0 = (P^- - P^+)M_{11} + (Q^+ + Q^-)M_{21} + Q_0 M_{31} - P.$$

Family of conformal maps

$$\omega(\zeta) = -\frac{i\mu_0}{\tau_1}\Phi_2^+(\zeta, u), \quad (\zeta, u) \in \mathcal{D}^+.$$

There are three model parameters: $\hat{\tau}_1 = \tau_1/\mu_0$, $\hat{\tau}_1^\infty = \tau_1^\infty/\mu_0$, and $\kappa = \mu_1/\mu_0$.

In addition, the map $\hat{\omega}(\zeta) = N_1^{-1}\omega(\zeta)$ has $n+2$ real parameters, $\hat{N}_0^* = N_0^*/N_1$, $\hat{b}_0 = b_0/N_1$, and n geometric parameters say, m, k_3, $k_4, \ldots k_{n+1}$.

$N_1 \neq 0$ is a scaling parameter.

The other $n-1$ geometric parameters $k_{n+2}, \ldots, k_{2n}$ are determined by a system of $n-1$ nonlinear equations which are the conditions at infinity.

In the elliptic case, there are no nonlinear equations. In addition to the scaling parameter N_1, there is only one free geometric parameter, m. The map is invariant of the parameter $\hat{b}_0$, while $\hat{N}_0^*$ is a translation parameter.

Verification of the solution

For $(\xi^\pm, u^\pm) \in l_0^\pm \cup l_1^\pm \subset \mathcal{D}^+$,

$$\omega(\xi^\pm) = -\frac{i\mu_0}{\hat{\tau}_1} X^+(\xi^\pm, u^\pm)[\Psi(\xi^\pm, u^\pm) + \Omega(\xi^\pm, u^\pm)],$$

$u^\pm = p^{1/2}(\xi^\pm)$.

Contour l_0. On the two sides of the contour $l_0^\pm$, the functions $\Psi(\xi^\pm, u^\pm)$ and $\Omega(\xi^\pm, u^\pm)$ are real.

By the Sokhotski-Plemelj formulas, $\operatorname{Re} X^+(\xi^\pm, u^\pm) = 0$.

Therefore $\operatorname{Im}\omega(\xi^\pm) = 0$, $\operatorname{Re}\omega(\xi^\pm) \in (-\infty, \infty)$. This means that the contour l_0 is mapped onto the real axis of the physical plane.

Contour l_1. On applying the Sokhotski-Plemelj formulas,

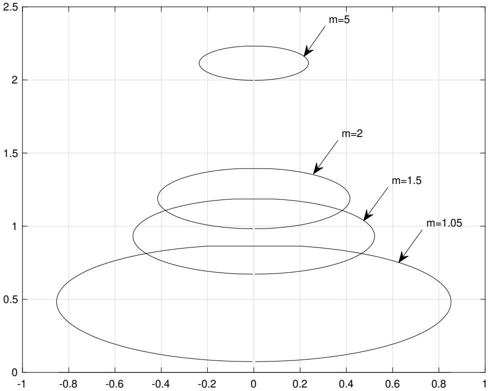
$$\Phi_1^+(\xi, v) - \Phi_1^-(\xi, v) = 2ib_1, \quad (\xi, v) \in l_1,$$

$$\Phi_2^+(\xi, v) - \Phi_2^-(\xi, v) = 2i\lambda_1[\operatorname{Re}\Phi_1^+(\xi, v) - a_1], \quad (\xi, v) \in l_1.$$

The complex boundary condition and the interface conditions hold. The solution satisfies the condition at infinity.

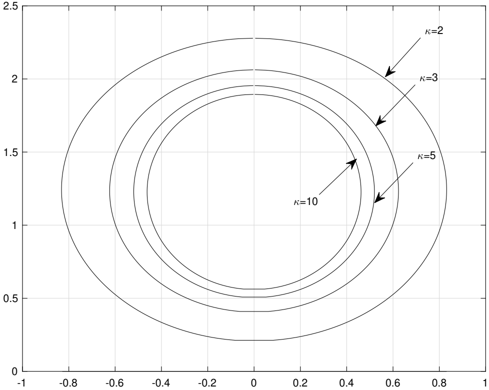
Uniformly stressed inclusion in a half-plane - 1

Normalized inclusion in the half-plane $|x_1| < \infty$, $x_2 \geq 0$ for different values of m when $\kappa = \mu_1/\mu_0 = 0.5$, $N_0^* = 0$, $\tau_1/\mu_0 = -1$, $\tau_1^\infty/\mu_0 = -2$.



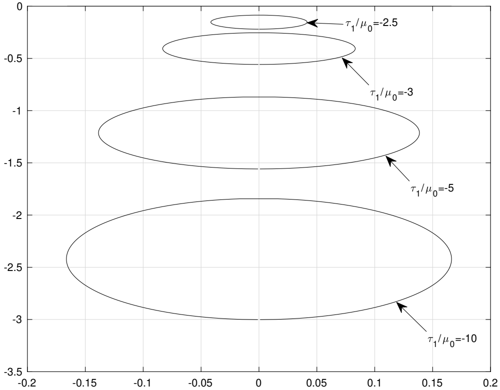
Uniformly stressed inclusion in a half-plane - 3

Normalized inclusion in the half-plane $|x_1| < \infty$, $x_2 \geq 0$ for different values of $\kappa \in (0,1)$ when $m = 1.6$, $N_0^* = 0$, $\tau_1/\mu_0 = -1$, $\tau_1^\infty/\mu_0 = -2$.



Uniformly stressed inclusion in a half-plane - 4

Normalized inclusion in the half-plane $|x_1| < \infty$, $x_2 \leq 0$ for different values of τ_1/μ_0 when $m = 2$, $N_0^* = 0$, $\kappa = 0.5$, $\tau_1^\infty/\mu_0 = -2$.



Model 2: A vortex patch in a wedge. (*A joint work with Anna Zemlyanova, Kansas State: to appear in Complex Variables Elliptic Equations)*

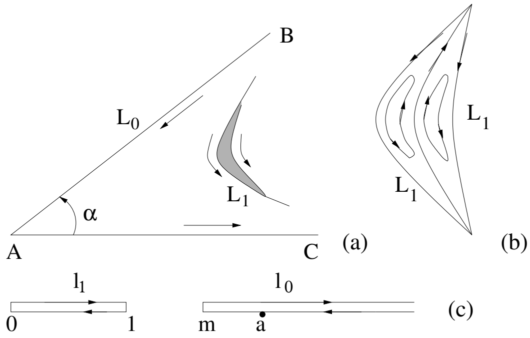


Figure: (a): Potential flow in a wedge around a vortex. (b): Mechanism of flow in the vortex $\mathcal{P}$. (c): Preimages of the vortex and the wedge boundary in the parametric plane.

Setting of Model 2

Let $\mathcal{W}$ be a wedge, $\mathcal{P} \subset \mathcal{W}$ be a bounded simply-connected domain whose boundary L_1 is not prescribed *a priori*, and $\tilde{\mathcal{D}} = \mathcal{W} \setminus \mathcal{P}$. Given two real constants Γ and U, recover the domain $\mathcal{P}$ and find a multi-valued function $w(z)$ with a cyclic period Γ,

$$\int_{L_1} \frac{dw}{dz} dz = \Gamma,$$

analytic in the domain $\tilde{\mathcal{D}}$,

$$w(z) \sim \frac{\Gamma}{2\pi i} \log z + C z^{\beta}, \quad z \to \infty, \quad z \in \tilde{\mathcal{D}},$$

On the boundary of the wedge $\mathcal{W}$,

$$\operatorname{Im} w = c_0, \quad z \in L_0, \quad \arg \frac{dw}{dz} = \begin{cases} \pi - \alpha, & z \in AB, \\ 0, & z \in AC. \end{cases}$$

On the boundary of the domain $\mathcal{P}$,

$$\operatorname{Im} w = c_1, \quad \left| \frac{dw}{dz} \right| = U, \quad z \in L_1.$$

Conformal mapping

Denote $l_0 = [m, +\infty)$ $(m > 1)$ and $l_1 = [0, 1]$. Let $z = f(\zeta)$ be a conformal map of a ζ-plane cut along these two segments onto the doubly connected flow domain $\tilde{\mathcal{D}}$.
$f(\zeta)$ maps l_0 and l_1 onto the wedge and vortex boundaries L_0 and L_1, Chaplygin representation

$$\frac{df}{d\zeta} = \omega_0(\zeta) e^{-\omega_1(\zeta)},$$

$$\omega_0(\zeta) = \frac{dw}{Ud\zeta}, \quad \omega_1(\zeta) = \log \frac{dw}{Udz}.$$

The function $\omega_0(\zeta)$ satisfies the boundary condition
$\operatorname{Im} \omega_0(\xi) = 0, \quad \xi \in l_0 \cup l_1$, and has the form

$$\omega_0(\zeta) = \frac{i(N_0 + N_1\zeta)}{p^{1/2}(\zeta)}, \quad p(\zeta) = \zeta(1-\zeta)(\zeta - m).$$

N_0 and N_1 are arbitrary real constants, $p^{1/2}(\zeta) = i\sqrt{|p(\xi)|}$ as $\zeta = \xi + i0$, $\xi > m$.

Hilbert problem for $\omega_1(\zeta)$

$$\operatorname{Im}\omega_1(\zeta) = \begin{cases} \pi-\alpha, & \zeta\in l_0', \\ 0, & \zeta\in l_0'', \end{cases} \qquad \operatorname{Re}\omega_1(\zeta)=0, \quad \zeta\in l_1. \qquad (1)$$

Here, $l_0'=[a,+\infty)^-$, $l_0''=[m,a]^-\cup[m,+\infty)^+$, and the superscripts $+$ and $-$ indicate that the segments belong to the upper and lower sides of the contour l_0,

$$\omega_1(\zeta)\sim\frac{\pi-\alpha}{2\pi}\log\zeta, \quad \zeta\to\infty.$$

If $a\neq m$, then

$$\omega_1(\zeta)\sim\frac{\pi-\alpha}{\pi}\log(\zeta-a), \quad \zeta\to a, \quad \omega_1(\zeta)\sim const, \quad \zeta\to\bar{a}.$$

If $a=m$, then

$$\omega_1(\zeta)\sim\frac{\pi-\alpha}{2\pi}\log(\zeta-a), \quad \zeta\to a, \quad \omega_1(\zeta)\sim const, \quad \zeta\to\bar{a}. \qquad (2)$$

Functions on a torus

Let $\mathcal{R}$ be a genus-1 Riemann surface (a torus) of the algebraic equation

$$u^2 = p(\zeta), \quad p(\zeta) = \zeta(1-\zeta)(\zeta - m),$$

$$u = \begin{cases} p^{1/2}(\zeta), & (\zeta, u) \in \mathbb{C}_1, \\ -p^{1/2}(\zeta), & (\zeta, u) \in \mathbb{C}_2, \end{cases}$$

Introduce a function on the surface $\mathcal{R}$

$$\Phi(\zeta, u) = \begin{cases} -i\omega_1(\zeta), & (\zeta, u) \in \mathbb{C}_1, \\ i\,\overline{\omega_1(\bar{\zeta})}, & (\zeta, u) \in \mathbb{C}_2, \end{cases}$$

$$\overline{\Phi(\zeta_*, u_*)} = \Phi(\zeta, u), \quad (\zeta, u) \in \mathcal{R}, \quad (\zeta_*, u_*) = (\bar{\zeta}, -u(\bar{\zeta})).$$

$$\Phi^+(\xi, v) = -i\omega_1(\xi) = -i\,\mathrm{Re}\,\omega_1(\xi) + \mathrm{Im}\,\omega_1(\xi),$$

$$\Phi^-(\xi, v) = i\overline{\omega_1(\xi)} = i\,\mathrm{Re}\,\omega_1(\xi) + \mathrm{Im}\,\omega_1(\xi), \quad (\xi, v) \in \mathcal{L}. \quad (3)$$

Riemann-Hilbert problem on a torus

Find all symmetric functions $\Phi(\zeta,u)$, $\overline{\Phi(\zeta_*,u_*)}=\Phi(\zeta,u)$ *analytic in* $\mathcal{R}\setminus\mathcal{L}$, *Hölder-continuous up to the boundary* $\mathcal{L}$ *apart from the singular points* $\zeta=a$ and $\zeta=\infty$ *with the boundary values satisfying*

$$\Phi^+(\xi,v)=G(\xi,v)\Phi^-(\xi,v)+g(\xi,v),\quad (\xi,v)\in\mathcal{L},$$

$$G(\xi,v)=\begin{cases}-1, & (\xi,v)\in l_0\\ 1, & (\xi,v)\in l_1,\end{cases}\qquad g(\xi,v)=\begin{cases}2(\pi-\alpha), & (\xi,v)\in l_0',\\ 0, & (\xi,v)\in l_0'',\\ 0, & (\xi,v)\in l_1.\end{cases}$$

The function $\Phi(\zeta,u)$ *has a logarithmic singularity at the infinite point* (∞,∞),

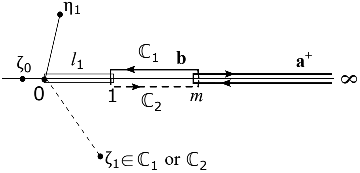
$$\Phi(\zeta,u)\sim\frac{\pi-\alpha}{2\pi i}\log\zeta,\quad \zeta\to\infty, \tag{4}$$

It is bounded at the point $(\bar a,u(\bar a))$ *and has a logarithmic singularity at the point* $(a,u(a))$,

$$\Phi(\zeta,u)\sim\frac{\pi-\alpha}{\sigma\pi i}\log(\zeta-a),\quad \zeta\to a, \tag{5}$$

where $\sigma=1$ *if* $a\neq m$ and $\sigma=2$ *otherwise.*

Solution of the Riemann-Hilbert problem on a torus



1. Analogue of the Cauchy kernel is $dV = \frac{1}{2}\left(\frac{\zeta-\zeta_0}{\xi-\zeta_0} + \frac{u}{v}\frac{\xi-\zeta_0}{\zeta-\zeta_0}\right)\frac{d\xi}{\xi-\zeta}$.
2. Factorization problem is

$$X^+(\xi,v) = G(\xi,v)X^-(\xi,v), \quad (\xi,v)\in\mathcal{L}\subset\mathcal{R}, \quad X(\zeta,u) = \overline{X(\bar{\zeta},-u(\bar{\zeta}))},$$

$$X(\zeta,u) = \exp\left\{\left(\frac{1}{2}-2n_a\right)\frac{u(\zeta)}{i(\zeta-\zeta_0)}\int_m^\infty \frac{(\xi-\zeta_0)d\xi}{\sqrt{|p(\xi)|}(\xi-\zeta)}\right.$$

$$-\frac{1}{2}\int_\gamma\left[\left(\frac{\zeta-\zeta_0}{\xi-\zeta_0}+\frac{u(\zeta)}{u(\xi)}\frac{\xi-\zeta_0}{\zeta-\zeta_0}\right)\frac{d\xi}{\xi-\zeta}+\left(\frac{\zeta-\zeta_0}{\bar{\xi}-\zeta_0}-\frac{u(\zeta)}{u(\bar{\xi})}\frac{\bar{\xi}-\zeta_0}{\zeta-\zeta_0}\right)\frac{d\bar{\xi}}{\bar{\xi}-\zeta}$$

3. Jacobi inversion problem.

Exact representation of the conformal mapping $z = f(\zeta)$

The conformal mapping is

$$f(\zeta) = i\int_a^\zeta \frac{N_0 + iN_1\xi}{p^{1/2}(\xi)} e^{-i\Phi^+(\xi,u(\xi))} d\xi, \quad \zeta \in \mathcal{D},$$

The solution has to be single-valued

$$\int_{l_1} \frac{df}{d\zeta} d\zeta = 0$$

and satisfy the circulation condition

$$\int_0^1 \frac{(N_0 + N_1\xi)d\xi}{\sqrt{|p(\xi)|}} = -\frac{\Gamma}{2U}.$$

This gives

$$N_0 = -\left[1 - \frac{\mathbf{K}(k) - \mathbf{E}(k)}{k^2\mathbf{K}(k)} \frac{C_0}{C_1}\right]^{-1} \frac{\Gamma}{4Uk\mathbf{K}(k)}, \quad N_1 = -\frac{N_0 C_0}{C_1}.$$

and the transcendental equation $C_0 S_1 - C_1 S_0 = 0$ for a.

Complex potential

$$C_j = \int_0^1 \frac{c^+(\xi) + c^-(\xi)}{\sqrt{|p(\xi)|}} \xi^j d\xi, \quad S_j = \int_0^1 \frac{s^+(\xi) + s^-(\xi)}{\sqrt{|p(\xi)|}} \xi^j d\xi, \quad j = 0, 1.$$

$$c^\pm(\xi) = \cos[\Phi^+(\xi, u^\pm(\xi))], \quad s^\pm(\xi) = \sin[\Phi^+(\xi, u^\pm(\xi))], \quad \xi \in l_1^\pm,$$

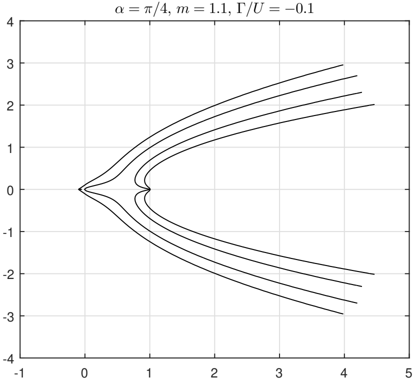
Let the transcendental equation with respect to the parameter a have a solution and it is unique. Suppose the image of the segment with the starting and terminal points $\zeta = m$ and $\zeta = 1$, respectively, does not intersect the contour L_1. Then Model 2 has a one-parametric family of solutions

$$w(z) = U \int_a^\zeta e^{i\Phi^+(\xi, u(\xi))} d\xi + const, \quad z = f(\zeta),$$

with m being a free parameter and $\Phi^+(\xi, u(\xi))$ being the solution of the Riemann-Hilbert problem on the first sheet $\mathbb{C}_1$ of the torus $\mathcal{R}$, while $z = f(\zeta)$ is the conformal map.

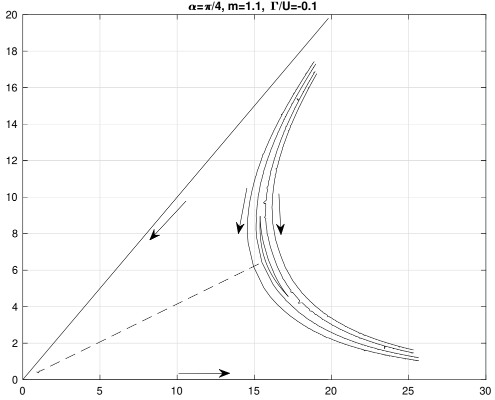
Preimages of streamlines

The preimages of some streamlines when $\alpha = \pi/4$, $\Gamma/U = -0.1$, $m = 1.1$.

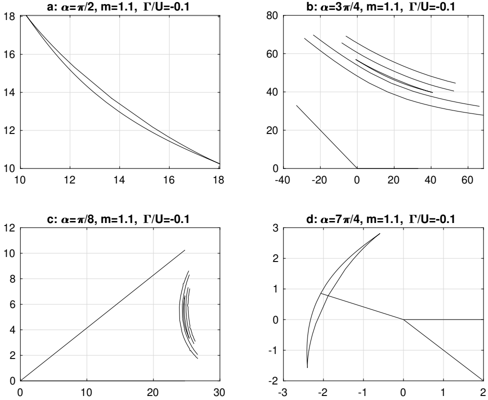


Vortex patch and streamlines - 1

The vortex domain and some streamlines when $\alpha = \pi/4$, $\Gamma/U = -0.1$, $m = 1.1$.



The vortex domain, the wedge, and some streamlines when (a) $\alpha = \frac{9}{10}\pi$, $m = 1.1$ (b) $\alpha = \frac{9}{10}\pi$, $m = 1.2$ (c) $\alpha = \frac{1}{3}\pi$, $m = 1.1$, d) $\alpha = \frac{1}{3}\pi$, $m = 1.5$.



Conclusions

- A closed-form solution to the inverse problem of antiplane shear of an elastic half-plane with n finite elastic inclusions has been derived. The boundary of the half-plane is kept free of traction and the stress field inside the inclusions is uniform, while the shape of the inclusions is to be recovered.
- An n-parametric family of conformal mappings has been obtained by the method of the Riemann-Hilbert problem on a hypereliptic surface.
- A new analogue for a hyperelliptic surface has been proposed and studied. Factorization function has been recovered by employing this kernel and solving the associated Jacobi inversion problem. The solution has been explicitly written in the elliptic ($n = 1$) case.
- A free boundary roblem of a vortex patch in a wedge has been solved by the method of conformal mappings and the Riemann-Hilbert problem on a torus. It has been shown that the vortex patch has two cusps.