Control and Inverse Problems for Krein's String

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Initial Boundary Value Problem – Krein's String

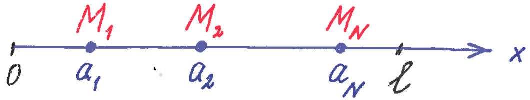
$$\rho(x)\,\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad \rho \in \text{ piecewise } C^2[0,\ell],\ \rho(x) > 0$$

$$t \in (0,T),\ x \in \Omega := (0,a_1) \cup (a_1,a_2) \cup \ldots \cup (a_N,\ell),$$

$$u(a_j^-,t) = u(a_j^+,t), \quad M_j\, u_{tt}(a_j,t) = u_x(a_j^+,t) - u_x(a_j^-,t),$$

$$u(x,0) = u_t(x,0) = 0,$$

$$u(0,t) = f(t),\ f \in \mathcal{F}^T := L^2(0,T), \quad u(\ell,t) = 0.$$



Regularity and Compatibility Conditions

We show that the wave transmitted through a mass is more regular than the incoming wave. We define the spaces W, W^T :

$$W = \{\phi \in L^2(0, a_1) \times H^1(a_1, a_2) \times \ldots \times H^N(a_N, \ell) :$$

$$\phi(a_j^-) = \phi(a_j^+),\ \phi'(a_j^-) = \phi'(a_j^+) - M_j \phi''(a_j^+)/\rho(a_j^+),\ \phi(\ell) = 0\Big\},$$

$$W^T = \{\phi \in W : \phi(x) = 0 \text{ for } x \geq X(T), \}$$

where

$$T = \int_0^{X(T)} \sqrt{\rho(x)}\, dx\,, \quad L = \int_0^{l} \sqrt{\rho(x)}\, dx\,.$$

Regularity and Controllability

Theorem

Suppose $T \leq L := \int_0^l \sqrt{\rho(x)}\, dx$. *For any* $f \in \mathcal{F}^T$, $u^f \in C(0,T;W^T)$ *and for any* $\phi \in W^T$, *there exists a unique* $f \in W^T$ *such that* $u^f(x,T) = \phi(x)$. *Furthermore,*

$$\|u^f(\cdot,T)\|_W \asymp \|f\|_{\mathcal{F}^T}.$$

For $T > L$,

$$\{u^f(\cdot,T) : f \in L^2(0,T)\} = W.$$

For $T > 2L$,

$$\left\{\left(u^f(\cdot,T),\, u_t^f(\cdot,T)\right) : f \in L^2(0,T)\right\} = W \times W_{-1},$$

$$W_{-1} := H^{-1}(0,a_1) \times L^2(a_1,a_2) \times H^1(a_2,a_3) \times \ldots \times H^{N-1}(a_N,\ell).$$

Spectral Problem

$$-\varphi''(x,\lambda) = \lambda^2 \rho(x)\varphi(x,\lambda), \quad x \in \Omega,$$
$$\varphi(0,\lambda) = \varphi(\ell,\lambda) = 0, \quad \varphi(a_j^-,\lambda) = \varphi(a_j^+,\lambda),$$
$$-M_j\,\lambda^2 \varphi(a_j,\lambda) = \varphi'(a_j^+,\lambda) - \varphi'(a_j^-,\lambda) \;\; \forall j.$$

The eigenvalues λ_n^2 of this problem are simple and the eigenfunctions φ_n form the orthonormal basis in the space $\mathcal{H} := L^2_\rho(0,l) \oplus \mathbb{R}^N$ with inner product

$$\langle \phi, \psi \rangle_{\mathcal{H}} = \int_0^l \phi(x)\psi(x)\rho(x)dx + \sum_{j=1}^{N} M_j \phi(a_j)\psi(a_j).$$

The inverse spectral data (SD) is the set $\{\lambda_n^2, \varphi_n'(0)\}$, $n \in \mathbb{N}$.
Titchmarsh–Weyl function: $\mathcal{M}(\lambda)\,\phi(0,\lambda) = \phi'(0,\lambda)$.

Dynamical Inverse Problem

Response operator

$$R^T : \mathcal{F}^T \mapsto \mathcal{F}^T, \quad Dom(R^T) = \{f \in H^1(0,T), f(0) = 0\},$$

$$(R^T f)(t) = u_x^f(0,t), \ t \in (0,T).$$

Theorem

Let $T > 2L$. Given R^T, one can find $\rho(x), \ a_j, \ M_j, \ l$.

Connecting Operator

$$C^T : \mathcal{F}^T \mapsto \mathcal{F}^T, \quad (C^T f, g)_{\mathcal{F}^T} := \left\langle u^f(\cdot,T), u^g(\cdot,T)\right\rangle_{\mathcal{H}} .$$

The connecting operator can be written in the form $C^T = (U^T)^*(U^T)$ where

$$U^T : \mathcal{F}^T \mapsto W^T, \; U^T f = u^f(\cdot,T).$$

The exact controllability theorem implies that $Ker\, C^T = \emptyset$.

Theorem

Operator C^T can be explicitly expressed through the response operator on the double interval: $C^T = -\frac{1}{2}(S^T)^* I^{2T} R^{2T} S^T$, where

$$(S^T f)(t) = \begin{cases} f(t) & \text{if} \quad t \in [0,T], \\ -f(2T-t) & \text{if} \quad t \in (T,2T], \end{cases} \quad (I^{2T} f)(t) = \int_0^t f(s)ds.$$

Sketch of the proof: Set $w(s,t) := \langle u^f(\cdot,s), u^g(\cdot,t)\rangle_{\mathcal{H}}$.
We notice that $(C^T f, g)_{\mathcal{F}^T} = w(T,T)$.

$$w_{tt}(s,t) - w_{ss}(s,t) = \int_0^l [u^f(x,s)u^g_{tt}(x,t) - u^f_{ss}(x,s)u^g(x,t)]\rho(x)dx$$

$$+\sum_j M_j[u^f(a_j,s)u^g_{tt}(a_j,t) - u^f_{ss}(a_j,s)u^g(a_j,t)] = (\text{using } \rho u_{tt} = u_{xx})$$

$$= \left[u^f(x,s)u^g_x(x,t) - u^f_x(x,s)u^g(x,t)\right]_{x=0}^l = (Rf)(s)g(t) - f(s)(Rg)(t).$$

We use $w(s,0) = w_t(s,0) = w(0,t) = 0$ to find $w(T,T)$
by D'Alembert's formula.

We set

$$\mathcal{H}^T = \{\phi \in \mathcal{H} : \phi(x) = 0 \text{ for } x \geq X(T)\}.$$

Bases in $\mathcal{F}^T$ and $\mathcal{H}^T$

Let $T \le L$ and $\{f_n\}$, $n \in \mathbb{N}$, be a basis in $\mathcal{F}^T$ such that

$$f \in C^2[0,T],\, f(0) = f'(0) = 0, \quad \left(C^T f_k, f_n\right)_{\mathcal{F}^T} = \delta_{kn}.$$

Due to controllability, $\{u^{f_n}(\cdot,T)\}$ is an orthonormal basis in $\mathcal{H}^T$.

We introduce two functions: $\phi^0(x) = 1$, $\phi^1(x) = x$, $x \in [0,l]$ and let ϕ^0_T and ϕ^1_T be their restrictions to the interval $[0, X(T)]$.

Theorem

The coefficients in the series representations of the functions ϕ^0_T, ϕ^1_T with respect to the basis $\{u^{f_j}(\cdot,T)\}$ have the form

$$c^0_n := \langle \phi^0, u^{f_n}(\cdot,T)\rangle_{\mathcal{H}} = -\int_0^T (T-t)(R^T f_n)(t)\, dt\,,$$

$$c^1_n := \langle \phi^1, u^{f_n}(\cdot,T)\rangle_{\mathcal{H}} = \int_0^T (T-t) f_n(t)\, dt\,.$$

Proof

Sketch of the proof:

$$\langle \phi^0, u^{f_n}(\cdot, T)\rangle_{\mathcal{H}} = \int_0^l u^{f_n}(x,T)\rho(x)dx + \sum_j M_j u^{f_n}(a_j, T)$$

$$= \int_0^T (T-t)\left[\int_0^{X(T)} u_{tt}^{f_n}(x,t)\rho(x)dx + \sum_j M_j u_{tt}^{f_n}(a_j,t)\right] dt$$

$$= \int_0^T (T-t)\left[\int_0^{X(T)} u_{xx}^{f_n}(x,t)dx + \sum_j M_j u_{tt}^{f_n}(a_j,t)\right] dt$$

$$= -\int_0^T (T-t)u_x^{f_n}(0,t)\,dt = -\int_0^T (T-t)(R^T f_n)(t)\,dt.$$

Solution of the Dynamical Inverse Problem

We introduce two functions:

$$\mu(T) = \int_0^{X(T)} \rho(x)dx + \sum_{j:\ a_j<X(T)} M_j\,,$$

$$\nu(T) = \int_0^{X(T)} x\rho(x)dx + \sum_{j:\ a_j<X(T)} M_j\, a_j\,.$$

They can be found using the theorem:

$$\mu(T) = \langle \phi^0_T, \phi^0_T \rangle_{\mathcal{H}} = \sum_n |c^0_n|^2, \quad \nu(T) = \langle \phi^0_T, \phi^1_T \rangle_{\mathcal{H}} = \sum_n c^0_n\, c^1_n\,.$$

Separating the singular and regular (integral) parts, we find M_j and a_j from the singular parts. From the regular parts we have

$$\dot{\mu}_r(T) = \rho(X(T))\,\dot{X}(T), \quad \dot{\nu}_r(T) = X(T)\rho(X(T))\,\dot{X}(T)\,.$$

From these relations we find $X(T)$ and, finally, $\rho(x)$.

A Tree

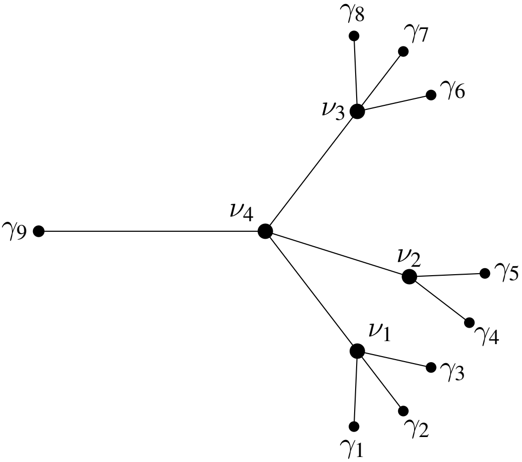


Fig. 2: A metric tree

Network of Strings with Attached Masses

We assume that $\rho(x)$ is a positive C^2 function defined on each edge and a positive mass M_ν is attached to each internal vertex:

$$\rho(x)\, u_{tt} - u_{xx} = 0 \ \text{ in } \{\Gamma \setminus V\} \times (0,T) \tag{1}$$

$$\begin{aligned} &\textstyle\sum_{e_j \sim \nu} \partial u_j(\nu,t) = M_\nu u_{tt}(\nu,t) \text{ at each vertex } \nu \in V \setminus \partial\Gamma \\ &u(\cdot,t) \text{ is continuous at each vertex, for } t \in [0,T] \end{aligned} \tag{2}$$

$$u = f \quad \text{on} \ \ \partial\Gamma \times [0,T], \ \ u|_{t=0} = u_t|_{t=0} = 0 \ \text{ in } \ \Gamma \tag{3}$$

In (2) (and below) $\partial u_j(\nu)$ denotes the derivative of u at the vertex ν taken along the edge e_j in the direction outwards the vertex. Also, $e_j \sim \nu$ means edge e_j is incident to vertex ν, and the sum is taken over all edges incident to ν.

Solution of the IBVP

We show that the wave transmitted through a mass is more regular than the incoming wave.

Let d_i be a combinatorial distance from edge e_i to the boundary $\partial\Gamma$ and $\mathcal{H} = L^2(\Gamma)$, $\mathcal{F}^T = L^2([0,T];\mathbb{R}^m)$.

Theorem

If $f \in \mathcal{F}^T$ then the (generalized) solution u^f of our IBVP exists, unique and for any $t \in [0,T]$, $u^f(\cdot,t) \in \mathcal{H}$ and $u^f \in C([0,T];\mathcal{H})$. Furthermore, $u^f|_{e_i} \in C([0,T];H^{d_i}(e_i))$.

Controllability

We present now our results on control and inverse problems for trees.

Let $d = \max\ d_i$ and $diam\,\Gamma := \max_{i \neq j} dist\{\gamma_i, \gamma_j\}$.

Theorem

For any $T \geq diam\,\Gamma$ and any $y \in \mathcal{H}^d$, $z \in \mathcal{H}^{d-1}$, there exists $f \in \mathcal{F}^T$ such that $u^f(\cdot, T) = y$ and $u_t^f(\cdot, T) = z$.

Dynamical Inverse Problem

We define the *response operator*, $R^T = \{R^T_{ij}\}_{i,j=1}^m$, on $\mathcal{F}^T$ by

$$(R^T f)(t) = \partial u^f(\cdot, t)|_{\partial\Gamma} , \qquad 0 < t < T . \tag{4}$$

The next theorem describes the solution of the dynamical inverse problem.

Theorem

The operator R^T known for $T > diam\,\Gamma$ uniquely determines the graph topology, the lengths of the edges $\{l_j : j = 1, \ldots, N\}$, the masses $\{M_\nu : \nu \in V \setminus \partial\Gamma\}$, and the potential q on Γ. If the topology is known, all other parameters can be found from the main diagonal $\{R^T_{ii}\}_{i=1}^m$ of the response operator.

The inverse problem can be solved using the reduced response operator, $\{R^T_{ij}\}_{i,j=1}^{m-1}$, known for

$$T > T_m := 2 \max_{j=1,\ldots,m-1} dist\,(\gamma_m, \gamma_j).$$

The Leaf Peeling Method

Our algorithm allows us to recalculate the response operator from the original graph to a smaller graph by “pruning" boundary edges. Ultimately, it allows us to reduce the original inverse problem on the graph to the inverse problem on a single interval.

Knowledge of R_{jj}^{T} for sufficiently large T allows to recover the length of the edge $e \in E$ incident to γ_j, and the potential ρ on e. It also allows to recover the total number of edges incident to ν, where $\nu \in V \setminus \partial\Gamma$ is an internal vertex to which e is incident.

The proof of these statements uses the analysis of the waves incoming to, transmitted through and reflected from vertex ν and is based on the Boundary Control method in inverse theory.

Spectrum

$$-\varphi''(x,\lambda) = \lambda^2 \rho(x)\,\varphi(x,\lambda),$$

$$x \in \Omega = (0,a_1) \cup (a_1,a_2) \cup \ldots \cup (a_N,\ell),$$

$$\varphi(0,\lambda) = \varphi(\ell,\lambda) = 0, \;\; \varphi(a_j^-,\lambda) = \varphi(a_j^+,\lambda),$$

$$-M_j\,\lambda^2 \varphi(a_j,\lambda) = \varphi'(a_j^+,\lambda) - \varphi'(a_j^-,\lambda),$$

The (simple) eigenfrequences λ_n of this problem are zeros of the generating function

$$F(\lambda) := \Phi(\ell,\lambda), \;\; \Phi(0,\lambda) = 0, \; \Phi'(0,\lambda) = 1.$$

They can be presented as a union of $N+1$ separated series:

$$\{\lambda_n\}_{n=1}^{\infty} = \{\lambda_m^0\}_{m=1}^{\infty} \cup \left[\bigcup_{j=1}^{N} \{\lambda_m^j\}_{m=0}^{\infty}\right], \;\; \lambda_m^j = \frac{\pi m}{L_j} + O\left(\frac{1}{m}\right).$$

Fourier Method

We present the solution of the IBVP in the form of the series

$$u(x,t)=\sum_{n=1}^{\infty} b_n(t)\,\phi_n(x),\ \ u_t(x,t)=\sum_{n=1}^{\infty} \dot{b}_n(t)\,\phi_n(x).$$

Coefficients $b_n(t)$ can be found as

$$b_n(t)=\phi_n'(0)\int_0^t f(\tau)\,\frac{\sin\lambda_n(t-\tau)}{\lambda_n}\,d\tau,$$

$$\dot{b}_n(t)=\phi_n'(0)\int_0^t f(\tau)\cos\lambda_n(t-\tau)\,d\tau.$$

These equalities can be conveniently written in the form

$$c_n^{\pm}(t)=\int_0^t f(\tau)\,e^{\pm i\lambda_n\tau}\,d\tau,\ \ c_n^{\pm}(t):=\left(\mp\frac{i\lambda_n}{\phi_n'(0)}\,b_n(t)+\frac{\dot{b}_n(t)}{\phi_n'(0)}\right)\,e^{i\lambda_n t}.$$

Fourier Method

We use the spectral representation to create a scale of spaces $\mathcal{H}_p$ (for $p > 0, \ \mathcal{H}_p = D(A^{p/2})$:

$$\mathcal{H}_p = \{\varphi(x) = \sum_{n=1}^{\infty} \alpha_n \phi_n(x) \ : \ ||\varphi||_p^2 = \sum_{n=1}^{\infty} |\alpha_n|^2 |n|^{2p} < \infty\}, \ p \in \mathbb{R}.$$

So, for example,

$$\|u(\cdot,t)\|_{\mathcal{H}_0}^2 \asymp \sum_{n=1}^{\infty} |b_n(t)|^2, \ \|u_t(\cdot,t)\|_{\mathcal{H}_{-1}}^2 \asymp \sum_{n=1}^{\infty} |\dot{b}_n(t)|^2 \, |\lambda_n|^{-2}.$$

For a regular string, $|\lambda_n| \asymp n, \ |\phi_n'(0)| \asymp n$, and so

$$\|u(\cdot,t)\|_{L^2(0,\ell)}^2 + \|u_t(\cdot,t)\|_{H^{-1}(0,\ell)}^2 \asymp \sum_{n=1}^{\infty} |c_n^{\pm}(t)|^2.$$

The Moment Problem

Exact controllability of our system in $L^2(0,\ell) \times H^{-1}(0,\ell)$ in the time interval $[0,T]$ is equivalent to solvability of the following moment problem in $L^2(0,T)$ for all $\{c_n^\pm\} \in \ell^2$:

$$c_n^\pm = \int_0^T f(t)\, e^{\pm i\lambda_n t}\, d\tau,\ n \in \mathbb{N}.$$

The latter is in turn equivalent to the fact that the family $\{\exp(\pm i\lambda_n t)\}$ is a Riesz sequence in $L^2(0,T)$.
This is true (for a regular string) iff $T \geq 2L$.
The sharp regularity result is also follows from the Riesz sequence property.

For strings with attached masses:
(i) it is not clear how express non-symmetric spaces W_0 and W_1 in terms of the coefficients α_n;
(ii) the relation $|\phi_n'(0)| \asymp n$ is not generally true;
(iii) for no T the family $\{\exp(\pm i\lambda_n t)\}$ can be a Riesz sequence in $L^2(0,T)$ because the sequence $\{\lambda_n\}$ is not separated: $\inf_{n\neq k} |\lambda_n - \lambda_k| = 0$.

Assume $\{\mu_j\}$ is a non-repeating sequence. The exponential divided difference (EDD) of order zero for $\{e^{i\mu_n t}\}$ is $[e^{i\mu_1 t}](t) := e^{i\mu_1 t}$. The EDD of order $n-1$ is given by

$$[e^{i\mu_1 t}, \dots, e^{i\mu_n t}] = \frac{[e^{i\mu_1 t}, \dots, e^{i\mu_{n-1} t}] - [e^{i\mu_2 t}, \dots, e^{i\mu_n t}]}{\mu_1 - \mu_n}, \ \mu_1 \neq \mu_n.$$

One then easily derives the formulas

$$[e^{i\mu_1 t}, e^{i\mu_2 t}] = \frac{e^{i\mu_1 t} - e^{i\mu_2 t}}{\mu_1 - \mu_2}, \quad [e^{i\mu_1 t}, \dots, e^{i\mu_n t}] = \sum_{k=1}^{n} \frac{e^{i\mu_k t}}{\prod_{j\neq k}(\mu_k - \mu_j)}.$$

Bases of Exponential Divided Differences

The Riesz basis theory of exponential divided differences was developed in (Avdonin and Ivanov, 2001; Avdonin and Moran, 2001).
The generating function of the sequence $\{\pm\lambda_n\}$ is an entire function

$$F(\lambda) = \prod_{n\in\mathbb{N}} \left(1 - \frac{\lambda^2}{\lambda_n^2}\right)$$

of the exponential type of L in $\mathbb{C}_\pm$. For some $h \in \mathbb{R}$,

$$|F(x+ih)| \asymp |x+ih|^{N-1}.$$

Therefore, the corresponding family of EDD $\mathcal{E} = \bigcup_{p\in\pm\mathbb{N}} \mathcal{E}_p$ forms a Riesz basis in the closure of its linear span in $L^2(0,T)$ for $T > 2L$.

The original moment problem

$$c_n^{\pm} = \int_0^T f(t)\, e^{\pm i\lambda_n t}\, d\tau,\ n \in \mathbb{N},$$

can be rewritten in the form

$$\hat{c}_n^{\pm} = \int_0^T f(t)\, e_n^{\pm}(t)\, d\tau,\ n \in \mathbb{N};\ \ B\, col\, \{e^{\pm i\lambda_n t}\} = col\, \{e_n^{\pm}(t)\},$$

where $\mathcal{E} = \{e_n^{\pm}(t)\}$ is the corresponding family of EDD.
Similarly, the relations

$$\alpha_n = \int_0^T f(t)\ \sin(\lambda_n t)\, d\tau,\ n \in \mathbb{N},$$

can be rewritten in the form

$$\beta_n = \int_0^T f(t)\, s_n(t)\, d\tau,\ n \in \mathbb{N};\ \ B\, col\, \{\sin(\lambda_n t)\} = \ col\, \{s_n(t)\}.$$

Shape Controllability

Since $\mathcal{E}$ forms a Riesz sequence in $L^2(0,T)$ for $T > 2L$, the family $S = \{s_n\}$ of the corresponding DD of sine functions forms a a Riesz sequence in $L^2(0,T)$ for $T > L$.
Now we rewrite the relation $y(x,T) = \sum \alpha_n \phi_n(x)$ in the form $y(x,T) = \sum \beta_n \psi_n(x)$, where ψ_n are the corresponding linear combinations of ϕ_n : $col\{\psi_n\} = (B^{-1})^* \, col\{\phi_n\}$.
Shape controllability. Let $T > L, \; y \in W_0$. There exists

$$f \in L^2(0,T), \; \|f\|^2_{L^2(0,T)} \asymp \|y\|^2_{W_0},$$

such that

$$u^f(x,T) = y(x), \; x \in \Omega.$$

This theorem implies that $\{\psi_n\}$ forms a Riesz basis in W_0. From that we can derive the full controllability for $T > 2L$.

References–I

S. Avdonin, N. Avdonina and J. Edward, *Boundary inverse problems for networks of vibrating strings with attached masses,* Proceedings of Dynamic Systems and Applications, **7** (2016), 1–4.

S.A. Avdonin, V.S. Mikhaylov and K.B. Nurtazina, *On inverse dynamical and spectral problems for the wave and Schrödinger equations on finite trees. The leaf peeling method,* Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov (POMI), (2015), **438**, 7–21.

S. Avdonin and S. Nicaise, *Source identification problems for the wave equation,* Inverse Problems, (2015), **31**, 095007, 29pp.

S. Avdonin and J. Bell, *Determining a distributed conductance parameter for a neuronal cable model defined on a tree graph,* Journal of Inverse Problems and Imaging, (2015), **9**, no. 3, 645-659.

References–II

S. Avdonin, A. Choque Rivero, G. Leugering and V. Mikhaylov, *On the inverse problem of the two velocity tree-like graph,* Zeit. Angew. Math. Mech., (2015), **95,** no. 12, 1490–1500.

S. Avdonin, B. Belinskiy and M. Matthews, *Dynamical inverse problems on a metric tree*, Inverse Problems, **27** (2011), 075011, 21pp.

S. Avdonin, P. Kurasov and M. Nowaczyk, *On the reconstruction of boundary conditions for star graphs*, Inverse Problems and Imaging, **4** (2010), no. 4, 579–598.

S. Avdonin, *Control problems on quantum graphs*, in: "Analysis on Graphs and Its Applications", Proceedings of Symposia in Pure Mathematics, AMS, **77** (2008), 507–521.

S. Avdonin and P. Kurasov, *Inverse problems for quantum trees,* Inverse Problems and Imaging, **2** (2008), no. 1, 1–21.