Norm inequalities for linear and multilinear singular integrals on weighted and variable exponent Hardy spaces

David V. Cruz-Uribe, OFS

University of Alabama

SFU Rostov, Russia

Acknowledgments

Joint work with:

- Kabe Moen
- Hanh Nguyen

Convolution-type singular integrals

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y) f(y)\, dy,$$

$$|\partial_x^\alpha K(x)| \leqslant \frac{A}{|x|^{n+|\alpha|}}, \qquad |\alpha| \leqslant N+1$$

$$T : L^2 \to L^2$$

Hardy space estimates

$0 < p \leqslant 1$, $f \in H^p$ if $\mathcal{M}_{N_0} f \in L^p$, where

$$\mathcal{M}_{N_0} f(x) = \sup_{\varphi \in \mathfrak{F}_{N_0}} \sup_{t>0} \big|\varphi_t * f(x)\big|.$$

Theorem (Fefferman, Stein, 1972)

If T is a convolution-type SIO, $N > \lfloor n(\frac{1}{p} - 1)\rfloor$, then

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Theorem (Fefferman, Stein, 1972)

If T is a convolution-type SIO, $N > \lfloor n(\frac{1}{p} - 1) \rfloor$, then

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Calderón-Zygmund SIOs

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,dy$$

$$|K(x,y)| \leqslant \frac{C}{|x-y|^n}, \quad x \neq y,$$

$$|K(x,y+h) - K(x,y)| + |K(x+h,y) - K(x,y)| \leqslant \frac{C|h|^\delta}{|x-y|^{n+\delta}}$$

for all $|h| \leqslant \frac{1}{2}|x-y|$, where $C > 0$ and $0 < \delta \leqslant 1$.

$$T : L^2 \to L^2$$

Hardy space estimates II

Theorem (Alvarez, Milman, 1986, DCU-KM-HN, 2018)

If T is a CZ-SIO, $0 < p \leqslant 1$, the kernel satisfies

$$|\partial_y^\beta K(x,y+h) - \partial_y^\beta K(x,y)| \leqslant \frac{C|h|^\delta}{|x-y|^{n+N+\delta}}, \quad |\beta| = N+1$$

and

$$\int x^\beta Ta(x)dx = 0$$

for all $(N+1,\infty)$ atoms a and $|\beta| \leqslant N$,

$$N > \left\lfloor n\left(\frac{1}{p}-1\right)\right\rfloor,$$

then $T : H^p \to H^p$.

Bilinear singular integrals

$$T(f_1, f_2)(x) = \int_{(\mathbb{R}^n)^2} K(x, y_1, y_2) f_1(y_1) f_2(y_2)\, dy_1 dy_2$$

$$\left|\partial_{y_0}^{\alpha_0} \partial_{y_1}^{\alpha_1} \partial_{y_2}^{\alpha_2} K(y_0, y_1, y_2)\right| \leqslant A\Big(\sum_{k,l=0}^{2} |y_k - y_l|\Big)^{-(2n+|\alpha_0|+|\alpha_1|+|\alpha_2|)}$$

for all $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ such that $|\alpha| = |\alpha_0| + |\alpha_1| + |\alpha_2| \leqslant N$,

$$T : L^{q_1}(\mathbb{R}^n) \times L^{q_2}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$$

for some $1 < q_1,\ q_2 < \infty$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$.

Bilinear singular integrals

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Bilinear Hardy estimates

Theorem (Grafakos, Kalton, 2001)

If T is a bilinear SIO, $0 < p,\, p_1,\, p_2 \leqslant 1$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, *and*

$$N > \left\lfloor n\left(\frac{1}{p} - 1\right)\right\rfloor,$$

then $T : H^{p_1} \times H^{p_2} \to L^p$.

Goal

Generalize bilinear theorem of Grafakos and Kalton to

- weighted Hardy spaces
- variable exponent Hardy spaces

Weights

A **weight** w is a non-negative, locally integrable function.

Simple example: $w(x) = |x|^a$, $a \in \mathbb{R}$.

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx \right)^{1/p}.$$

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Muckenhoupt A_p weights

For $1 < p < \infty$, $w \in A_p$ if

$$\sup_Q \fint_Q w\,dx \left(\fint_Q w^{1-p'}\,dx \right)^{p-1} < \infty.$$

When $p = 1$, $w \in A_1$ if

$$\fint_Q w\,dx \leqslant C \operatorname*{ess\,inf}_{x \in Q} w(x).$$

For $1 < p < q < \infty$, $A_1 \subset A_p \subset A_q$.

$r_w = \inf\{p : w \in A_p\}, \qquad A_\infty = \bigcup_{p \geqslant 1} A_p.$

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Hardy-Littlewood maximal operator

$$Mf(x) = \sup_Q \fint_Q |f(y)|\,dy \cdot \chi_Q(x)$$

Theorem (Muckenhoupt 1972)

For $1 < p < \infty$, TFAE:

- $w \in A_p$
- $M : L^p(w) \to L^p(w)$

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Weighted Hardy spaces

For $0 < p < \infty$, $w \in A_\infty$, $f \in H^p(w)$ if $\mathcal{M}_{N_0} f \in L^p(w)$.

If $p > 1$ and $w \in A_p$, $H^p(w) = L^p(w)$.

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Variable exponents

$\mathcal{P}_0(\mathbb{R}^n)$ set of $p(\cdot) : \mathbb{R}^n \to (0, \infty)$.

$$p_- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \qquad p_+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x).$$

$p(\cdot) \in LH$ if

$$|p(x) - p(y)| \leqslant \frac{C_0}{-\log(|x-y|)}, \qquad 0 < |x-y| < \frac{1}{2},$$

and

$$|p(x) - p_\infty| \leqslant \frac{C_\infty}{\log(e + |x|)}.$$

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Variable Lebesgue spaces

$f \in L^{p(\cdot)}$ if

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leqslant 1\right\} < \infty.$$

If $p(\cdot) = p$, then $L^{p(\cdot)} = L^p$.

If $p(\cdot) \in LH$ and $p_- > 1$,

$$M : L^{p(\cdot)} \to L^{p(\cdot)}.$$

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Weighted estimates

Theorem (DCU-KM-HN, 2018)

If T is a bilinear SIO, $0 < p,\ p_1,\ p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $w_1,\ w_2 \in A_\infty$, $w = w_1^{p/p_1} w_2^{p/p_2}$, and

$$N > \max\left\{\left\lfloor 2n\left(\frac{r_{w_1}}{p_1} - 1\right)\right\rfloor, \left\lfloor 2n\left(\frac{r_{w_2}}{p_2} - 1\right)\right\rfloor\right\} + n,$$

then $T : H^{p_1}(w_1) \times H^{p_2}(w_2) \to L^p(w)$.

Variable exponent estimates

Theorem (DCU-KM-HN, 2018)

If T is a bilinear SIO, $p(\cdot)$, $p_1(\cdot)$, $p_2(\cdot) \in \mathcal{P}_0$, $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, and

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then $T : H^{p_1(\cdot)} \times H^{p_2(\cdot)} \to L^{p(\cdot)}$.

The miracle of RdF extrapolation

Variable exponent theorem is consequence of weighted theorem and bilinear extrapolation:

Theorem (DCU-Naibo, 2016)

Suppose $0 < p, p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $w_1, w_2 \in A_1$, $w = w_1^{p/p_1} w_2^{p/p_2}$, *and*

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Then for $q(\cdot), q_1(\cdot), q_2(\cdot) \in \mathcal{P}_0$, $\frac{1}{q(\cdot)} = \frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)}$, $p_1 < (q_1)_-$, $p_2 < (q_2)_-$,

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Proof of weighted theorem

Proof of weighted theorem depends on five pieces:

- Finite atomic decomposition
- local/global decomposition
- Pointwise kernel estimates
- Vector-valued inequalities
- Rubio de Francia extrapolation

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Weighted Hardy space estimates

Theorem (DCU-KM-HN, 2018; Hart & Olivera 2017)

Given $w \in A_\infty$, if T is a CZ-SIO, $0 < p < \infty$, the kernel satisfies

$$|\partial_y^\beta K(x, y+h) - \partial_y^\beta K(x,y)| \leqslant \frac{C|h|^\delta}{|x-y|^{n+N+\delta}}, \quad |\beta| = N+1$$

and

$$\int x^\beta Ta(x)dx = 0$$

for all $(N+1, \infty)$ atoms a and $|\beta| \leqslant N$,

$$N > \left\lfloor n\left(\frac{r_w}{p} - 1\right)\right\rfloor,$$

then $T : H^p(w) \to H^p(w)$.

Key ideas in proof

Finite atomic decomposition: dense set $f \in H^p(w)$ such that

$$f = \sum_{i=1}^{M} \lambda_i a_i$$

with $(N+1, \infty)$ atoms $\{a_i\}$ with supports $\{Q_i\}$, $\lambda_i > 0$, and

$$\|f\|_{H^p(w)} \approx \left\| \sum_{i=1}^{M} \lambda_i \chi_{Q_i} \right\|_{L^p(w)}.$$

$$Tf(x) = \sum_{i=1}^{M} \lambda_i Ta_i(x).$$

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Equivalent norm

$$\|f\|_{H^p(w)} \approx \|M_\phi f\|_{L^p(w)}$$

where

$$M_\phi f(x) = \sup_{t>0} |\phi_t * f(x)|,$$

$\phi \in C_c^\infty$, $\mathrm{supp}(\phi) \subset B(0,1)$,

$\int \phi \, dx = 1$, $\phi_t(x) = t^{-n}\phi(x/t)$.

Local and global decomposition

$$\begin{aligned}\|Tf\|_{H^p(w)} &\approx \|M_\phi Tf\|_{L^p(w)} \\ &\leqslant \left\| \sum_{i=1}^{M} \lambda_i M_\phi T(a_i)\chi_{Q_i^*} \right\|_{L^p(w)} \\ &\qquad + \left\| \sum_{i=1}^{M} \lambda_i M_\phi T(a_i)\chi_{(Q_i^*)^c} \right\|_{L^p(w)} \\ &= \mathit{Local} + \mathit{Global}.\end{aligned}$$

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Pointwise estimate away from support

Lemma (Folklore)

If a is an $(N+1,\infty)$ atom with support in Q, then for all $x \in (Q^*)^c$,

$$M_\phi(Ta)(x) \leqslant CM(\chi_Q)(x)^{\frac{n+N+1}{n}}.$$

A vector-valued inequality

Lemma (Fefferman-Stein, 1971, Andersen-John, 1980)

For $1 < p < \infty$, $1 < r < \infty$, $w \in A_p$,

$$\bigg\| \Big(\sum_i (Mf_i)^r \Big)^{1/r} \bigg\|_{L^p(w)} \leqslant C \bigg\| \Big(\sum_i |f_i|^r \Big)^{1/r} \bigg\|_{L^p(w)},$$

Vector-valued inequality is an **immediate** consequence of scalar inequality by Rubio de Francia extrapolation.

Global estimate

For $w \in A_\infty$, if $\tau = \frac{n+N+1}{n}$, then $w \in A_{\tau p}$:

$$Global \lesssim \left\| \sum_{i=1}^{M} \lambda_i M(\chi_{Q_i})^\tau \right\|_{L^p(w)}$$

$$= \left\| \left(\sum_{i=1}^{M} \lambda_i M(\chi_{Q_i})^\tau \right)^{\frac{1}{\tau}} \right\|_{L^{p\tau}(w)}^{\tau}$$

$$\lesssim \left\| \left(\sum_{i=1}^{M} \lambda_i \chi_{Q_i} \right)^{\frac{1}{\tau}} \right\|_{L^{p\tau}(w)}^{\tau}$$

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Another vector-valued estimate

Lemma (Grafakos-Kalton, 2001, DCU-KM-HN, 2018)

Fix $q > 1$. For all $0 < p < q$ and $w \in RH_{(q/p)'}$ and every non-negative sequence $\{g_k\}$ such that $\mathrm{supp}(g_k) \subset Q_k$,

$$\Big\| \sum_k g_k \Big\|_{L^p(w)} \leqslant C \Big\| \sum_k \Big(\fint_{Q_k} g_k^q \, dx \Big)^{1/q} \chi_{Q_k} \Big\|_{L^p(w)}.$$

Hard proof by G-K when $q = 1$. Very easy proof using reverse Hölder extrapolation due to T. Anderson-DCU-KM (2018).

Local estimate

If $w \in A_\infty$, there exists $q > 1$ so that $w \in RH_{(q/p)'}$. Then by L^q estimates:

$$Local \lesssim \left\| \sum_{i=1}^{M} \lambda_i \left(\fint_{Q_i} M_\phi T(a_i)^q \, dx \right)^{\frac{1}{q}} \chi_{Q_i^*} \right\|_{L^p(w)}$$

$$\lesssim \left\| \sum_{i=1}^{M} \lambda_i \left(\fint_{Q_i} |a_i|^q \, dx \right)^{\frac{1}{q}} \chi_{Q_i^*} \right\|_{L^p(w)}$$

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$$
\begin{aligned}
Local &\lesssim \left\| \sum_{i=1}^{M} \lambda_i \left(\fint_{Q_i} M_\phi T(a_i)^q \, dx \right)^{\frac{1}{q}} \chi_{Q_i^*} \right\|_{L^p(w)} \\
&\lesssim \left\| \sum_{i=1}^{M} \lambda_i \left(\fint_{Q_i} |a_i|^q \, dx \right)^{\frac{1}{q}} \chi_{Q_i^*} \right\|_{L^p(w)} \\
&\lesssim \left\| \sum_{i=1}^{M} \lambda_i \chi_{Q_i^*} \right\|_{L^p(w)} \\
&\lesssim \left\| \sum_{i=1}^{M} \lambda_i M(\chi_{Q_i})^\tau \right\|_{L^p(w)}.
\end{aligned}
$$

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2) multilinear multipliers with weak regularity

3) multilinear fractional singular integrals

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Two open questions

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Спасибо!

Thank You!

Roll Tide!