FEWNOMIALS IN L^1: GEOMETRY OF THE UNIT SPHERE

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Introduction

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- A *lacunary polynomial*, or *fewnomial*, in $\mathcal{P}_N$ is (roughly) a polynomial therein that has "gaps", i.e., is spanned by some selected monomials from the family $\{z^k : k = 0, \dots, N\}$ rather than by all of them.

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- Suppose that N and $k_1, \dots, k_M$ are positive integers with

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and let

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$$\Lambda := \{0, \dots, N\} \setminus \{k_1, \dots, k_M\}.$$

- Define $\mathcal{P}(\Lambda)$ as the space of polynomials of the form $\sum_{k \in \Lambda} c_k z^k$, with $c_k \in \mathbb{C}$.

- We shall restrict our polynomials to the circle

$$\mathbb{T} := \{\zeta \in \mathbb{C} : |\zeta| = 1\}$$

and embed $\mathcal{P}(\Lambda)$ in $L^1 = L^1(\mathbb{T})$, with norm

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- In particular, $\mathcal{P}(\Lambda) \subset H^1$, where

$$H^1 := \{f \in L^1 : \operatorname{spec} f \subset \mathbb{Z}_+\}.$$

- Now let $X = (X, \|\cdot\|)$ be a general Banach space, and let

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- Also, a point $\xi \in \mathrm{ball}(X)$ is said to be *exposed* for the ball if $\exists \phi \in X^*$ with $\|\phi\| = 1$ such that the set $\{x \in \mathrm{ball}(X) : \phi(x) = 1\}$ equals $\{\xi\}$.

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- **Problem:** Characterize the extreme/exposed points of the unit ball in $\mathcal{P}(\Lambda)$, viewed as a subspace of $L^1 = L^1(\mathbb{T})$.

Background results

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- ... not so easy to describe.
- Now let $\varphi \in L^\infty(\mathbb{T})$, and put

$$K_1(\varphi) := \{f \in H^1 : \overline{z\varphi f} \in H^1\}.$$

Assume that $K_1(\varphi) \neq \{0\}$. (In particular, this happens if $\varphi \equiv 0$ or if $\varphi = \overline{\theta}$, with θ inner.)

- The following result (K.D., last millennium) generalizes the de Leeuw–Rudin theorem.

Theorem A

For a function $f \in K_1(\varphi)$ *with* $\|f\|_1 = 1$, *TFAE:*
(i.A) f *is an extreme point of* $\mathrm{ball}(K_1(\varphi))$.
(ii.A) *The inner factors of* f *and* $\overline{z\varphi f}$ *are relatively prime.*

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- When $\varphi = \overline{z}^{N+1}$, we have $K_1(\varphi) = \mathcal{P}_N$. We write then $f := p$ and note that the role of $\overline{z\varphi f}$ goes to the polynomial p^*, given by

$$p^*(z) := z^N \overline{p\,(1/\overline{z})}, \qquad z \in \mathbb{C} \setminus \{0\}.$$

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- Equivalently, if $p(z) = \sum_{k=0}^{N} c_k z^k$, then $p^*(z) = \sum_{k=0}^{N} \overline{c}_k z^{N-k}$.

- Thus we arrive at

Theorem B

Suppose that $p \in \mathcal{P}_N$ and $\|p\|_1 = 1$. TFAE:
(i.B) *p is an extreme point of $\mathrm{ball}(\mathcal{P}_N)$.*
(ii.B) *The polynomials p and p^* have no common zeros in $\mathbb{D}$.*

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- **Remark:** Condition $(\mathrm{ii.B})$ means that p has no pair of symmetric zeros w. r. t. $\mathbb{T}$ and $|\widehat{p}(0)| + |\widehat{p}(N)| \neq 0$.

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- **Remark:** Condition (ii.B) means that p has no pair of symmetric zeros w. r. t. $\mathbb{T}$ and $|\widehat{p}(0)| + |\widehat{p}(N)| \neq 0$.
- The exposed points in $\mathcal{P}_N$ have also been determined (K.D., 2000):

Theorem C

Suppose that $p \in \mathcal{P}_N$ and $\|p\|_1 = 1$. TFAE:
(i.C) *p is an exposed point of* $\mathrm{ball}(\mathcal{P}_N)$.
(ii.C) *Condition* (ii.B) *is fulfilled, and p has no multiple zeros on $\mathbb{T}$.*

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- In particular, we can no longer divide polynomials by their elementary factors.
- **Example:** The polynomial $z \mapsto 1 - z^N$ is pretty lacunary, but

$$\frac{1 - z^N}{1 - z} = 1 + z + \cdots + z^{N-1}$$

is not.

- Recall that

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- Thus, the interesting case is that where condition (ii.B) fails, meaning that p and p^* *do* have common zeros in $\mathbb{D}$.
- To determine whether a unit-norm polynomial p from $\mathcal{P}(\Lambda)$ is an extreme point of the unit ball, we first cook up a certain matrix $\mathfrak{M} = \mathfrak{M}(p)$ from it.

Extreme points of $\mathrm{ball}(\mathcal{P}(\Lambda))$: constructing the matrix $\mathfrak{M}(p)$

- Given $p \in \mathcal{P}(\Lambda)$ with $\|p\|_1 = 1$, recall that $p^*(\in \mathcal{P}_N)$ is defined by

$$p^*(z) := z^N \overline{p\,(1/\overline{z})}, \qquad z \in \mathbb{C} \setminus \{0\}.$$

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- Now let $a_1, \dots, a_n$ be the (distinct) common zeros of p and p^* lying in $\mathbb{D}$, and let

$$m_j := \min \{\mathrm{mult}(a_j, p), \mathrm{mult}(a_j, p^*)\}, \qquad j = 1, \dots, n.$$

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- Also, put

$$m := \sum_{j=1}^{n} m_j.$$

- Further, consider the polynomial

$$G(z) := \prod_{j=1}^{n} (z - a_j)^{m_j} (1 - \overline{a}_j z)^{m_j},$$

which divides both p and p^*, and the ratio

$$R(z) := p(z)/G(z),$$

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- Then write

$$A(k) := \mathrm{Re}\, \widehat{R}(k), \qquad B(k) := \mathrm{Im}\, \widehat{R}(k) \qquad (k \in \mathbb{Z}).$$

- Now define, for $j = 1, \dots, M$ and $\ell = 0, \dots, m$, the numbers

$$A^+_{j,\ell} := A(k_j+\ell-m)+A(k_j-\ell-m), \quad B^+_{j,\ell} := B(k_j+\ell-m)+B(k_j-\ell-m),$$

$$A^-_{j,\ell} := A(k_j+\ell-m)-A(k_j-\ell-m), \quad B^-_{j,\ell} := B(k_j+\ell-m)-B(k_j-\ell-m).$$

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- From these, we build the $M \times (m+1)$ matrices

$$\mathcal{A}^+ := \left\{A^+_{j,\ell}\right\}, \qquad \mathcal{B}^+ := \left\{B^+_{j,\ell}\right\}$$

and the $M \times m$ matrices (with j as above and $\ell = 1, \dots, m$)

$$\mathcal{A}^- := \left\{A^-_{j,\ell}\right\}, \qquad \mathcal{B}^- := \left\{B^-_{j,\ell}\right\}.$$

- Finally, we need the block matrix

$$\mathfrak{M} = \mathfrak{M}(p) := \begin{pmatrix} \mathcal{A}^+ & \mathcal{B}^- \\ \mathcal{B}^+ & -\mathcal{A}^- \end{pmatrix},$$

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- Our main result $\#1$ now says:

Theorem

Suppose that $p \in \mathcal{P}(\Lambda)$ and $\|p\|_1 = 1$. Then p is an extreme point of $\mathrm{ball}(\mathcal{P}(\Lambda))$ if and only if $\operatorname{rank} \mathfrak{M} = 2m$.

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- Our main result #2 describes the exposed points of $\mathrm{ball}(\mathcal{P}(\Lambda))$ in similar terms.

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- Our main result #2 describes the exposed points of $\mathrm{ball}(\mathcal{P}(\Lambda))$ in similar terms.
- Well, not *quite* similar, actually...

The lacunary case: exposed points of $\mathrm{ball}(\mathcal{P}(\Lambda))$

- Recall, once again, that

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- Thus, the interesting case is that where condition (ii.C) fails, meaning that either p and p^* have common zeros in $\mathbb{D}$, or p has multiple zeros on $\mathbb{T}$.

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- Consequently, whenever a polynomial $p \in \mathcal{P}(\Lambda)$ is an exposed point of $\mathrm{ball}(\mathcal{P}_N)$, it will be exposed for $\mathrm{ball}(\mathcal{P}(\Lambda))$.
- Thus, the interesting case is that where condition (ii.C) fails, meaning that either p and p^* have common zeros in $\mathbb{D}$, or p has multiple zeros on $\mathbb{T}$.
- Once again, our criterion for a unit-norm polynomial $p \in \mathcal{P}(\Lambda)$ to be an exposed point of $\mathrm{ball}(\mathcal{P}(\Lambda))$ will be stated in terms of a certain matrix $\widetilde{\mathfrak{M}} = \widetilde{\mathfrak{M}}(p)$ built from it.

- Given $p \in \mathcal{P}(\Lambda)$ with $\|p\|_1 = 1$, let $\zeta_1, \dots, \zeta_\nu$ be the distinct zeros of p lying on $\mathbb{T}$, and let $\lambda_1, \dots, \lambda_\nu$ be their respective multiplicities.

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- Put

$$\mu_j := [\lambda_j/2], \qquad j = 1, \dots, \nu,$$

where $[\cdot]$ denotes integral part, and

$$\mu := \sum_{j=1}^{\nu} \mu_j, \qquad \widetilde{m} := m + \mu.$$

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- Now define

$$G_0(z) := \prod_{j=1}^{\nu} (z - \zeta_j)^{\mu_j} (1 - \overline{\zeta}_j z)^{\mu_j}.$$

- Further, consider the polynomials

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- Finally, let $\widetilde{\mathfrak{M}} = \widetilde{\mathfrak{M}}(p)$ be the block matrix built from the coefficients of $\widetilde{R}$ in exactly the same way as $\mathfrak{M} = \mathfrak{M}(p)$ was built from those of R.

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- This matrix $\widetilde{\mathfrak{M}}$ has $2M$ rows and $2\widetilde{m} + 1$ columns.

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- This matrix $\widetilde{\mathfrak{M}}$ has $2M$ rows and $2\widetilde{m}+1$ columns.

- Also, let

$$\widetilde{\mathcal{N}} := \ker \widetilde{\mathfrak{M}}$$

  be the null-space of the linear map $\widetilde{\mathfrak{M}} : \mathbb{R}^{2\widetilde{m}+1} \to \mathbb{R}^{2M}$.

Plus-vectors and plus-dimension

- Suppose $d \geq 0$ is an integer and

$$v := (\alpha_0, \alpha_1, \ldots, \alpha_d, \beta_1, \ldots, \beta_d)$$

is a vector in $\mathbb{R}^{2d+1}$.

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is a vector in $\mathbb{R}^{2d+1}$.

- We say that v is a *plus-vector* if

$$\alpha_0 + \sum_{k=1}^{d} (\alpha_k \cos kt - \beta_k \sin kt) \geq 0 \quad \text{for all} \;\; t \in (-\pi, \pi].$$

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$$\alpha_0 + \sum_{k=1}^{d} (\alpha_k \cos kt - \beta_k \sin kt) \geq 0 \quad \text{for all} \;\; t \in (-\pi, \pi].$$
- Finally, for a subspace $V \subset \mathbb{R}^{2d+1}$, we define its *plus-dimension* $\dim_+ V$ as the maximum number of linearly independent plus-vectors in V.

- **Remark:** An arbitrary *real-valued* trigonometric polynomial of degree d has the form

$$z \mapsto \mathrm{Re}\left\{\alpha_0 + \sum_{k=1}^{d}(\alpha_k + i\beta_k)z^k\right\}, \qquad z \in \mathbb{T},$$

where the α's and β's are real.

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- Now, *plus-vectors* are precisely the coefficient vectors of *nonnegative* trigonometric polynomials.

Exposed points of $\mathrm{ball}(\mathcal{P}(\Lambda))$: criterion

- Now, we have:

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Suppose that $p \in \mathcal{P}(\Lambda)$ and $\|p\|_1 = 1$. Then p is an exposed point of $\mathrm{ball}(\mathcal{P}(\Lambda))$ if and only if $\dim_+ \widetilde{\mathcal{N}} = 1$.

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- Yet another sufficient condition for p to be exposed is easy to state:

Proposition

If p is an extreme point of $\mathrm{ball}(\mathcal{P}(\Lambda))$ and if p has no multiple zeros on $\mathbb{T}$, then p is an exposed point of $\mathrm{ball}(\mathcal{P}(\Lambda))$.

Final remarks and open questions

- The reference is:

  K. M. Dyakonov, *Lacunary polynomials in L^1: geometry of the unit sphere*, Adv. Math. **381** (2021), 107607, 24 pp.

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- **A couple of open questions:**

  (1) What happens in higher dimensions (say, on $\mathbb{T}^d$ in place of $\mathbb{T}$)?

  (2) What about the Paley–Wiener type space

$$\left\{ f \in L^1(\mathbb{R}) : \operatorname{supp} \widehat{f} \subset \Lambda \right\},$$

  where Λ is a given compact subset of $\mathbb{R}$?

Last slide

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$\sim\sim\sim$ THANK YOU! $\sim\sim\sim$