Kirchhoff Laplacians on Metric Graphs

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(joint work with D. Mugnolo (Hagen) and N. Nicolussi (Paris))

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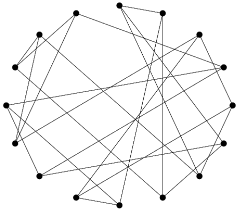
Combinatorial and Metric Graphs

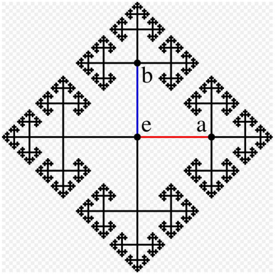
Definition

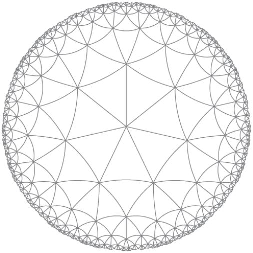
A graph is the set of vertices $\mathcal{V}$ and edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$.

For $u, v \in \mathcal{V}$, $u \sim v$ if there is an edge $e_{u,v} \in \mathcal{E}$ connecting u and v.
$\mathcal{E}_v := \{e_{u,v} \in \mathcal{E} \,|\, u \sim v\}$; the (combinatorial) degree (or valency) is

$$\deg(v) := \#\mathcal{E}_v, \quad v \in \mathcal{V}$$







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- $\mathcal{V}$ and $\mathcal{E}$ are at most countable
- $\mathcal{G}_d$ is simple, connected, and locally finite

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Definition (a.k.a. "cable graphs" or "metrized graphs")

If every edge $e \in \mathcal{E}$ is assigned with a positive finite length $|e| \in (0, \infty)$, then $\mathcal{G} = (\mathcal{V}, \mathcal{E}, |\cdot|)$ is called a metric graph

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  — degree 1 are "boundary" points,
- a non-Archimedean analog of Riemann surfaces
  a *tropical curve* or a degeneration of a smooth family of Riemann surfaces

Laplacians on Metric Graphs

Given $\mathcal{G} = (\mathcal{V}, \mathcal{E}, |\cdot|)$, identify each edge $e \in \mathcal{E}$ with $\mathcal{I}_e = [0, |e|]$,

$$L^2(\mathcal{G}) \cong \bigoplus_{e\in\mathcal{E}} L^2(e)$$

Kirchhoff Laplacian ("Laplace–Beltrami") Δ on $\mathcal{G}$ acts as $-\frac{\mathrm{d}^2}{\mathrm{dx}_e^2}$ on the interior of $\mathcal{G}$, and boundary conditions at the vertices:

Kirchhoff conditions at v: $\begin{cases} f \text{ is continuous at } v \\ \sum_{e\in\mathcal{E}_v} \partial_e f(v) = 0 \end{cases}$

The quantities

$$f_e(v) := \lim_{x_e\to v} f(x_e), \qquad \partial_e f(v) := \lim_{x_e\to v} \frac{f(x_e) - f_e(v)}{|x_e - v|},$$

are well defined for all $f \in \underbrace{H^2(\mathcal{G}\setminus\mathcal{V}) := \bigoplus_{e\in\mathcal{E}} H^2(e)}_{\text{Max. domain of distributional } \Delta \text{ in } L^2(\mathcal{G})}$

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- $\deg(v) = 1$: Kirchhoff = Neumann at v
- $\deg(v) = 2$: Kirchhoff = continuity of f and its derivative at v ("removable" singularity/inessential vertex)

Finite Metric Graphs: $\#\mathcal{V},\ \#\mathcal{E} < \infty$

$\mathcal{G}$ is compact iff finitely many vertices and edges.

- The Kirchhoff Laplacian Δ is self-adjoint in $L^2(\mathcal{G})$,
- Its spectrum is positive and discrete (resolvent is a compact operator),
- **Weyl's law**: Let $\sigma(\mathcal{G})$ be the "k-spectrum" ("wave spectrum") of Δ, ($k \in \sigma(\mathcal{G}) \ \Leftrightarrow\ \Delta f = k^2 f$ for some $f \neq 0$).

$$\#\sigma(\Delta) \cap [-\lambda, \lambda] = \frac{2\lambda}{\pi} \underbrace{\sum_{e \in \mathcal{E}} |e|}_{=\mathrm{vol}(\mathcal{G})} \quad + \quad \mathcal{O}(1), \quad \lambda \to +\infty.$$

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- Can one hear the shape of a metric graph? Gutkin&Smilansky, ...
- Quantum ergodicity? Berkolaiko, Colin de Verdiére,...

G. Berkolaiko, P. Kuchment, *Introduction to Quantum Graphs*, AMS, 2013.

Finite Metric Graphs and 1D Crystalline Measures

Poisson Summation Formula: If $\mathcal{G} = \mathbb{T}$, then $\sigma(\Delta_{\mathbb{T}}) = \mathbb{Z}$ and

$$\sum_{k\in\sigma(\Delta_{\mathbb{T}})} \delta_k = \widehat{\sum_{k\in\sigma(\Delta_{\mathbb{T}})} \delta_k}.$$

Problem (popularized by I. Meyer)

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Definition (Crystalline Measures/Fourier Quasi-Crystals)

$\mu = \sum_{x\in X} a_x\delta_x$ is a *crystalline measure* if it is a tempered distribution with $\widehat{\mu} = \sum_{\lambda\in\Lambda} b_\lambda\delta_\lambda$ such that X, Λ are discrete subsets of $\mathbb{R}$.
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CMs which are not Dirac combs (i.e., X is not an arithmetic progression) by Guinand (Acta, 1959), Meyer (2016), Lev & Olevskii (2016), . . .

F. Dyson, *Birds & Frogs*, Notices AMS, 2009.

Finite Metric Graphs and 1D Crystalline Measures

$$\mu_{\mathcal{G}} = \sum_{k\in\sigma(\Delta_{\mathcal{G}})} \delta_k \quad \Rightarrow \quad \widehat{\mu_{\mathcal{G}}}(t) = \sum_{k\in\sigma(\Delta_{\mathcal{G}})} \mathrm{e}^{-\mathrm{i}kt} = \mathrm{tr}(2\cos(\sqrt{\Delta}t)).$$

"baby" Selberg trace formula (Roth'1983; Kottos&Smilansky'1999)

$$\widehat{\mu_G} = \frac{2}{\pi}\mathrm{vol}(\mathcal{G})\delta_0 + \frac{1}{\pi}\sum_{p\in P}\ell(\mathrm{prim}\, p)\Big[s(p)\delta_{\ell(p)} + s(p)^*\delta_{-\ell(p)}\Big].$$

P periodic paths; $\mathrm{prim}\, p$ primitive path of p; $\ell(\cdot)$ length; s scattering coeff.
$\mathrm{supp}(\widehat{\mu}) \subseteq \Lambda = \{\sum_{e\in\mathcal{E}} n_e|e| : \ n_e \in \mathbb{Z}_{\geq 0}\}$ discrete set.

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Theorem (Kurasov & Sarnak'2020)

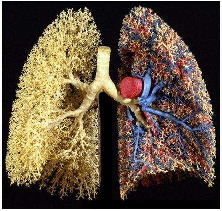
$\mu_{\mathcal{G}}$ is a positive FQC with uniformly discrete supp, generically not a Dirac comb.

Additive structure of $\sigma(\Delta)$ via (heavy) Diophantine analysis;

- YouTube: P. Sarnak, *Spectra of metric graphs and CMs*, IAS, feb.2020

Infinite Metric Graphs

- Further applications: "thin wire materials" in physics/biology/...



lungs $\approx$ *binary tree of 20-23 generations*
approx. $2 \times 10^6 - 1.6 \times 10^7$ *vertices*

Cast of human lungs (photo by E. Weibel)

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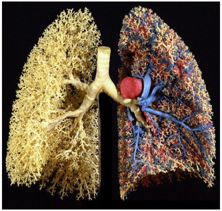

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N. Varopoulos, Long range estimates for Markov chains, *Bull. Sci. Math.* **109**, *(1985).*

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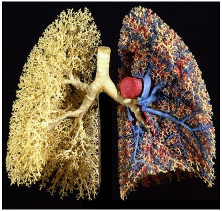

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- Connections: Geometric/topological group theory *(Cayley graphs)*
  Algebraic geometry *("singular Riemann surfaces")*

Kirchhoff conditions at v: $\begin{cases} f \text{ is continuous at } v \\ \sum_{e\in\mathcal{E}_v} \partial_e f(v) = 0 \end{cases}$

Definition:

The **maximal Kirchhoff Laplacian** H is defined on the domain

$$\mathrm{dom}(\Delta_{\mathrm{Kir}}) = \{ f \in H^2(\mathcal{G} \setminus \mathcal{V}) |\ (\text{Kirchhoff}) \text{ on } \mathcal{V} \}$$

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Problem: Do we need a boundary condition at "infinity"?

When $\Delta^0_{\mathrm{Kir}} = \Delta_{\mathrm{Kir}}$?
How to parameterize self-adjoint extensions?

Δ^0_{Kir} is symmetric, $(\Delta^0_{\mathrm{Kir}})^* = \Delta_{\mathrm{Kir}}$; Δ^0_{Kir} is self-adjoint $\Leftrightarrow \Delta^0_{\mathrm{Kir}} = \Delta_{\mathrm{Kir}}$.

Dirichlet, Neumann and Gaffney Laplacians

Quadratic form (Energy form/Dirichlet integral)

$$\mathfrak{t}[f] := \int_{\mathcal{G}} |\nabla f|^2 dx \quad \Big(= \langle \Delta f, f\rangle_{L^2} \quad \text{for } f \in \mathrm{dom}(\Delta^0_{\mathrm{Kir}}) \Big)$$

$\mathfrak{t}_N := \mathfrak{t} \restriction_{H^1(\mathcal{G})}$, where $H^1(\mathcal{G}) := H^1(\mathcal{G} \setminus \mathcal{V}) \cap C(\mathcal{G})$.

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The Dirichlet Δ_D and Neumann Δ_N Laplacians are the operators associated with, respectively, $\mathfrak{t}_D$ and $\mathfrak{t}_N$ in $L^2(\mathcal{G})$.

Dirichlet = absorbing boundary at "∞", Neumann = reflecting boundary at "∞"

- Δ_D is the Friedrichs extension, $\mathrm{dom}(\Delta_D) = \mathrm{dom}(\Delta_{\mathrm{Kir}}) \cap H^1_0(\mathcal{G})$.

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M.P.Gaffney, *The harmonic operator for exterior differential forms*, Proc. Natl. Acad. Sci. USA **37**, 48–50 (1951).

$\Delta_G = \Delta_N$ only if $\Delta_D = \Delta_N$. Otherwise, Δ_G is not symmetric!

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$A = A^* \geq 0$ is Markovian if e^{-tA} is positivity preserving, L^∞ contractive

- $\Delta_D = \Delta_N \;\Leftrightarrow$
- $H^1 = H^1_0$ (a.k.a. *$\mathcal{G}$ has neglecting boundary*) $\;\Leftrightarrow$
- $\Delta_G = \Delta_G^*$

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$\mathfrak{t}_D := \mathfrak{t} \restriction_{H^1_0(\mathcal{G})}$, where $H^1_0(\mathcal{G}) = \overline{H^1(\mathcal{G} \setminus \mathcal{V}) \cap C_c(\mathcal{G})}^{\|\cdot\|_{H^1}}$.

The Dirichlet Δ_D and Neumann Δ_N Laplacians are the operators associated with, respectively, $\mathfrak{t}_D$ and $\mathfrak{t}_N$ in $L^2(\mathcal{G})$.

$A = A^* \geq 0$ is Markovian if e^{-tA} is positivity preserving, L^∞ contractive

- $\Delta_D = \Delta_N \;\Leftrightarrow$
- $H^1 = H^1_0$ (a.k.a. *$\mathcal{G}$ has neglecting boundary*) $\;\Leftrightarrow$
- $\Delta_G = \Delta_G^* \;\Leftrightarrow\; \Delta^0_{\mathrm{Kir}}$ admits a unique Markovian extension.

Problem: When $H^1 = H^1_0$? parameterize Markovian extensions?

Self-adjont/Markovian Uniqueness via Harmonic Functions

Definition

$f \in C(\mathcal{G})$ is harmonic if $\Delta f = 0$/ λ-harmonic if $(\Delta + \lambda) f = 0$.

von Neumann formulas

$$\mathrm{dom}(\Delta_{\mathrm{Kir}}) = \mathrm{dom}(\Delta_D) \dot{+} \ker(\Delta_{\mathrm{Kir}} - \lambda), \quad \lambda \in \mathbb{C} \setminus \sigma(\Delta_D).$$

Since $\ker(\Delta_{\mathrm{Kir}} + \lambda) = L^2$ λ-harmonic functions and $\sigma(\Delta_D) \subseteq [0, \infty)$:

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Graph Boundaries

Poisson = bounded harmonic; Martin = positive harmonic, ...

Self-adjointness problem: What is known?

Natural path metric (geodesic/intrinsic distance) ρ on $\mathcal{G}$

$\varrho(u, v) = \inf_{\mathcal{P}=\{v_0,\dots,v_n\}:\ u=v_0,\ v=v_n} \sum_k |e_{v_{k-1},v_k}|.$

Gaffney-type Theorem:

If $(\mathcal{G}, \varrho)$ is complete, then $\Delta^0_{\mathrm{Kir}} = \Delta_D = \Delta_N = \Delta_G = \Delta_{\mathrm{Kir}}$.

For manifolds: Cauchy boundary $\partial_C M = \overline{M} \setminus M$
completeness, $\partial_C M = \emptyset \Rightarrow$ neglecting boundary (Gaffney, *Ann. of Math.* 1954)
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Proof:

Assume the converse: $\exists\ u \in L^2(\mathcal{G})$ such that $u \neq 0$ is λ-harmonic, $\lambda > 0$.
However, $|u| \geq 0$ is subharmonic.
By a version of Yau's L^p-Liouville theorem for strongly local Dirichlet forms, $|u| \equiv const$ if $(\mathcal{G},\varrho)$ is complete. Since $u \in L^2$, $u \equiv 0$. Contradiction. $\square$

K.-T. Sturm, *Analysis on local Dirichlet spaces I*, Crelle's J. (1994).

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Hopf–Rinow-type Theorem:

$(\mathcal{G},\varrho)$ is complete $\Leftrightarrow$ $(\mathcal{G},\varrho)$ is geodesically complete.

In particular, the most common assumption in the QG community $\inf_{e\in\mathcal{E}} |e| > 0$ implies geodesic completeness and hence self-adjointness.

G. Berkolaiko, P. Kuchment, *Introduction to Quantum Graphs*, AMS, 2013.

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Both $\mathfrak{t}_D$ and $\mathfrak{t}_N$ are strongly local Dirichlet forms; ϱ is intrinsic and completeness is a standard requirement for further analysis.
However, $\partial_C\mathcal{G} = \emptyset$ is far from being optimal! One can replace ϱ by "stronger" metrics, e.g., by *star path metric* (replace $|e_{v_{k-1},v_k}|$ at the top by $\mathrm{vol}(\mathcal{E}_{v_k})$).

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Then $H^1_0(\mathcal{G}) \neq H^1(\mathcal{G})$! Hence $\Delta_D \neq \Delta_N$ and $\Delta^0_{\mathrm{Kir}} \neq \Delta_{\mathrm{Kir}}$.

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- How can one **describe the self-adjoint extensions**?
  ("Classify versions of Schrödinger/Wave equation")
- How can one **describe the Markovian extensions**?
  ("Classify Brownian motions")

Boundaries for Infinite Graphs: Graph Ends

Definition (Freudenthal, 1944; Halin, 1964):

A **ray** in $\mathcal{G}$ is an infinite path $\mathcal{R} = (v_n)_{n \geq 0}$ without self-intersections.
Two rays are **equivalent**, if they **cannot be separated** by cutting out a finite set of vertices

A graph end is an equivalence class of rays. The set of ends is $\mathfrak{C}(\mathcal{G})$.

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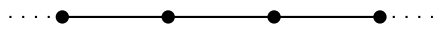

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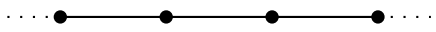
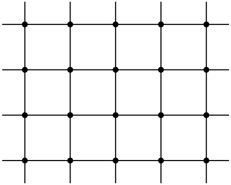


1 end



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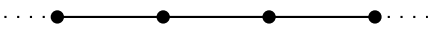

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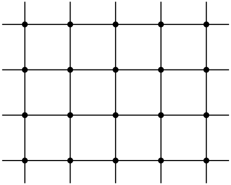
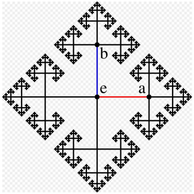


∞ many ends



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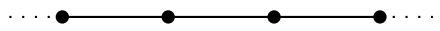

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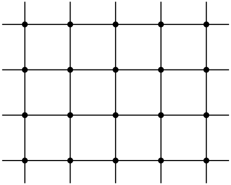

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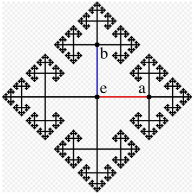


Theorem (Hopf–Freudenthal, 1940's; Stallings, 1968):

If $\mathcal{G}$ is a **Cayley graph** of a finitely generated countable group, then

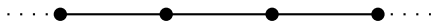
$$\#\mathfrak{C}(\mathcal{G}) \in \{1, 2, \infty\}.$$

Finite Volume Graph Ends

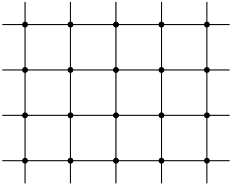
Definition

- A graph end γ has finite volume, if γ has a **neighborhood** U (w.r.t. the Freudenthal compactification) such that $\mathrm{vol}(U) < \infty$.
- The set of finite volume ends is denoted by $\mathfrak{C}_0(\mathcal{G})$.

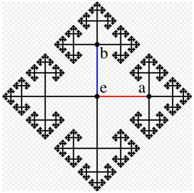
For the following examples, a graph end has finite volume, if...



... its ray has **finite length**



... the graph has **finite total volume**



... it has a **"subtree"** of finite total volume

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In particular, $\mathcal{G}$ has neglecting boundary, $H_0^1(\mathcal{G}) = H^1(\mathcal{G}) \iff \mathfrak{C}_0(\mathcal{G}) = \emptyset$.

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Graph ends and $H^1(\mathcal{G})$

H^1 is a subalgebra of $C_b(\mathcal{G})$ and $\mathfrak{C}_0(\mathcal{G})$ is a **proper boundary** for H^1 fn-s!

- Every $f \in H^1(\mathcal{G})$ has a **continuous extension** to $\mathcal{G} \cup \mathfrak{C}(\mathcal{G})$.
- $\mathfrak{C}_0(\mathcal{G})$ is an ideal boundary by **Gelfand theory**

Graph Ends and Markovian extensions

Corollary (AK–Mugnolo–Nicolussi'2019)

The following are equivalent:

- $\mathcal{G}$ has neglecting boundary, $H^1_0(\mathcal{G}) = H^1(\mathcal{G})$,
- Δ^0_{Kir} has a **unique Markovian extension**,
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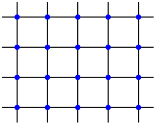
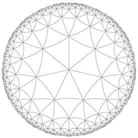
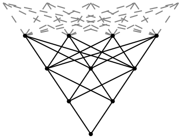
- If $\mathcal{G}$ has only one graph end:







Then **Markovian uniqueness** $\iff \mathrm{vol}(\mathcal{G}) = \sum_{e\in\mathcal{E}} |e| = \infty$

Description of Markovian Extensions

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(i) Self-adjoint restrictions of $\Delta_G \cong$ s.a. linear relations in $\ell^2(\mathfrak{C}_0(\mathcal{G}))$,

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- How to deal with **all self-adjoint extensions**?
  ... what are the **deficiency indices** $n_\pm(\Delta^0_{\mathrm{Kir}}) = \dim\ker(\Delta_{\mathrm{Kir}} \pm i)$?
  (von Neumann: s.a. extensions $\leftrightarrow$ unitary group $U(N)$, $N = n_\pm(\Delta^0_{\mathrm{Kir}})$)
  $n_\pm(\Delta^0_{\mathrm{Kir}}) \geq \#\mathfrak{C}_0(\mathcal{G})$. However, for Cayley graphs, $n_\pm(\Delta^0_{\mathrm{Kir}})$ do depend on the choice of a generating set! Even the case of $\mathbb{Z}$ is open...

Groups via Brownian motion on Metric Graphs?

Let Γ be a countable finitely generated group and $\mathcal{G}_d = (\Gamma, S)$ a Cayley graph w.r.t. some finite generating set $S = S^{-1}$.

Theorem (AK–Nicolussi// Calc. Var. (2019))

Γ is not amenable $\Leftrightarrow \lambda_0(\Delta_D) > 0$ for all $\mathcal{G} = (\mathcal{G}_d, |\cdot|)$ with $\sup_{\mathcal{E}} |e| < \infty$.

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$\#$ ends of Γ = dim(cone of Markovian restrictions of Δ_{Kir}) when $\mathrm{vol}(\mathcal{G}) < \infty$.

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Thank you for your attention!