Wiener algebras and trigonometric series in a coordinated fashion

E. Liflyand and R. Trigub

September, 2021

Synopsis

- We study Wiener algebras and Fourier series (of functions or measures) in their interrelation.

Synopsis

- We study Wiener algebras and Fourier series (of functions or measures) in their interrelation.
- Finding out many new connections between the two objects, one of which is non-periodic and the other is periodic, we show that they are much closer than it has been rated till recently.

Synopsis

- We study Wiener algebras and Fourier series (of functions or measures) in their interrelation.
- Finding out many new connections between the two objects, one of which is non-periodic and the other is periodic, we show that they are much closer than it has been rated till recently.
- This gives rise to a variety of new results, for which it is hardly understandable how to be on to them without these interrelations, as well as to new, much shorter proofs of some known ones.

Synopsis

- We study Wiener algebras and Fourier series (of functions or measures) in their interrelation.
- Finding out many new connections between the two objects, one of which is non-periodic and the other is periodic, we show that they are much closer than it has been rated till recently.
- This gives rise to a variety of new results, for which it is hardly understandable how to be on to them without these interrelations, as well as to new, much shorter proofs of some known ones.
- To proceed to more detailed analysis, we begin with basic definitions.

Synopsis

- We study Wiener algebras and Fourier series (of functions or measures) in their interrelation.
- Finding out many new connections between the two objects, one of which is non-periodic and the other is periodic, we show that they are much closer than it has been rated till recently.
- This gives rise to a variety of new results, for which it is hardly understandable how to be on to them without these interrelations, as well as to new, much shorter proofs of some known ones.
- To proceed to more detailed analysis, we begin with basic definitions.
- The main instance we are concerned in is the so-called Wiener algebras. We deal with the Wiener algebras of the following forms:

Wiener algebras

$$W_0 = W_0(\mathbb{R}) := \left\{ f(x) : \; f(x) = \int\limits_{\mathbb{R}} e^{ixy} g(y)\, dy, \; g \in L_1(\mathbb{R}) \right\}$$

with $\|f\|_{W_0} = \|g\|_{L_1}$;

$$W_1 = W_1(\mathbb{R}) := \left\{ f(x) : \; f(x) = c + f_0, \; f_0 \in W_0(\mathbb{R}), \; c \in \mathbb{C} \right\}$$

with $\|f\|_{W_1} = |c| + \|f_0\|_{W_0}$; and

$$W = W(\mathbb{R}) := \left\{ f(x) : \; f(x) = \int\limits_{\mathbb{R}} e^{ixy} d\mu(y), \mathrm{var}\mu < \infty \right\}$$

with $\|f\|_W = \mathrm{var}\mu$. By $W^+(\mathbb{R})$, we denote the subset of $W(\mathbb{R})$ with measure μ positive, that is, consisting of functions positive definite on $\mathbb{R}$.

$$W_0^*(\mathbb{R}) := \left\{ f(x) : f \in W_0(\mathbb{R}), \; \int_0^\infty \operatorname*{ess\,sup}_{|s| \ge t} |g(s)|\, dt < \infty \right\}.$$

Wiener algebras

- It is worth mentioning the classical Pólya's result on belonging to $W^+(\mathbb{R})$ of any even, bounded, convex and monotone decreasing to zero function on $[0, \infty)$.

Wiener algebras

- It is worth mentioning the classical Pólya's result on belonging to $W^+(\mathbb{R})$ of any even, bounded, convex and monotone decreasing to zero function on $[0, \infty)$.
- This implies the possibility of arbitrary decay to zero at infinity.

Wiener algebras

- It is worth mentioning the classical Pólya's result on belonging to $W^+(\mathbb{R})$ of any even, bounded, convex and monotone decreasing to zero function on $[0,\infty)$.
- This implies the possibility of arbitrary decay to zero at infinity.
- Moreover, such a function belongs to $W_0^*(\mathbb{R})$ as well.

Wiener algebras

- It is worth mentioning the classical Pólya's result on belonging to $W^+(\mathbb{R})$ of any even, bounded, convex and monotone decreasing to zero function on $[0, \infty)$.
- This implies the possibility of arbitrary decay to zero at infinity.
- Moreover, such a function belongs to $W_0^*(\mathbb{R})$ as well.
- A comprehensive survey of these algebras in L-Samko-Trigub (2012). $W_0^*(\mathbb{R})$ is studied in detail in Belinsky-L-Trigub (1997).

Wiener algebras

- It is worth mentioning the classical Pólya's result on belonging to $W^+(\mathbb{R})$ of any even, bounded, convex and monotone decreasing to zero function on $[0, \infty)$.
- This implies the possibility of arbitrary decay to zero at infinity.
- Moreover, such a function belongs to $W_0^*(\mathbb{R})$ as well.
- A comprehensive survey of these algebras in L-Samko-Trigub (2012). $W_0^*(\mathbb{R})$ is studied in detail in Belinsky-L-Trigub (1997).
- $W_0(\mathbb{R})$, $W_1(\mathbb{R})$ and $W(\mathbb{R})$ are Banach algebras with point-wise multiplication, and $W_0(\mathbb{R})$ is an ideal in $W(\mathbb{R})$.

Wiener algebras

- It is worth mentioning the classical Pólya's result on belonging to $W^+(\mathbb{R})$ of any even, bounded, convex and monotone decreasing to zero function on $[0,\infty)$.
- This implies the possibility of arbitrary decay to zero at infinity.
- Moreover, such a function belongs to $W_0^*(\mathbb{R})$ as well.
- A comprehensive survey of these algebras in L-Samko-Trigub (2012). $W_0^*(\mathbb{R})$ is studied in detail in Belinsky-L-Trigub (1997).
- $W_0(\mathbb{R})$, $W_1(\mathbb{R})$ and $W(\mathbb{R})$ are Banach algebras with point-wise multiplication, and $W_0(\mathbb{R})$ is an ideal in $W(\mathbb{R})$.
- $W_0(\mathbb{R})$ can be extended to $W_1(\mathbb{R})$ by adding the unity element to the former. All these algebras possess the local property.

Wiener algebras

- It is worth mentioning the classical Pólya's result on belonging to $W^+(\mathbb{R})$ of any even, bounded, convex and monotone decreasing to zero function on $[0, \infty)$.
- This implies the possibility of arbitrary decay to zero at infinity.
- Moreover, such a function belongs to $W_0^*(\mathbb{R})$ as well.
- A comprehensive survey of these algebras in L-Samko-Trigub (2012). $W_0^*(\mathbb{R})$ is studied in detail in Belinsky-L-Trigub (1997).
- $W_0(\mathbb{R})$, $W_1(\mathbb{R})$ and $W(\mathbb{R})$ are Banach algebras with point-wise multiplication, and $W_0(\mathbb{R})$ is an ideal in $W(\mathbb{R})$.
- $W_0(\mathbb{R})$ can be extended to $W_1(\mathbb{R})$ by adding the unity element to the former. All these algebras possess the local property.
- Note that $W_1(\mathbb{R})$ is a proper setting for the Wiener property of simultaneous belonging to the algebra of both $f \neq 0$ anywhere and $\frac{1}{f}$.

Trigonometric series

- Their periodic prototype, trigonometric series is a classical subject:

$$\sum_{k=-\infty}^{\infty} c_k e^{ikt}.$$

Trigonometric series

- Their periodic prototype, trigonometric series is a classical subject:

$$\sum_{k=-\infty}^{\infty} c_k e^{ikt}.$$

- We shall also use the classical (equivalent) form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

with the cosine coefficients a_n and the sine coefficients b_n.

Trigonometric series

- Their periodic prototype, trigonometric series is a classical subject:

$$\sum_{k=-\infty}^{\infty} c_k e^{ikt}.$$

- We shall also use the classical (equivalent) form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

  with the cosine coefficients a_n and the sine coefficients b_n.
- Under certain conditions on the sequence of coefficients $\{c_k\}$, such a series may be (or not be) the Fourier series of an integrable function, say f (written $f \in L_1(\mathbb{T})$, where $\mathbb{T} = [-\pi, \pi)$).

Trigonometric series

- Their periodic prototype, trigonometric series is a classical subject:

$$\sum_{k=-\infty}^{\infty} c_k e^{ikt}.$$

- We shall also use the classical (equivalent) form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

  with the cosine coefficients a_n and the sine coefficients b_n.
- Under certain conditions on the sequence of coefficients $\{c_k\}$, such a series may be (or not be) the Fourier series of an integrable function, say f (written $f \in L_1(\mathbb{T})$, where $\mathbb{T} = [-\pi, \pi)$).
- If yes, we shall say, with slight abuse of terminology, that the series is a Fourier series.

Trigonometric series

- In a wider setting, this series may be the Fourier-Stieltjes series of a Borel measure μ, or, equivalently, of a function of bounded variation F (written $V_{\mathbb{T}}(F) < \infty$; we shall write just $V(F)$ if it is clear on which set the total variation is taken).

Trigonometric series

- In a wider setting, this series may be the Fourier-Stieltjes series of a Borel measure μ, or, equivalently, of a function of bounded variation F (written $V_{\mathbb{T}}(F) < \infty$; we shall write just $V(F)$ if it is clear on which set the total variation is taken).
- In the former case the Fourier series is

$$f \sim \sum_{k=-\infty}^{\infty} \widehat{f}_k e^{ikt}, \qquad \widehat{f}_k = \frac{1}{2\pi} \int_{\mathbb{T}} f(t) e^{-ikt}\, dt,$$

Trigonometric series

- In a wider setting, this series may be the Fourier-Stieltjes series of a Borel measure μ, or, equivalently, of a function of bounded variation F (written $V_{\mathbb{T}}(F) < \infty$; we shall write just $V(F)$ if it is clear on which set the total variation is taken).
- In the former case the Fourier series is

$$f \sim \sum_{k=-\infty}^{\infty} \widehat{f}_k e^{ikt}, \qquad \widehat{f}_k = \frac{1}{2\pi} \int_{\mathbb{T}} f(t) e^{-ikt}\, dt,$$

- while in the latter case the Fourier-Stieltjes series is

$$dF \sim \sum_{k=-\infty}^{\infty} \widehat{dF}_k e^{ikt}, \qquad \widehat{dF}_k = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-ikt}\, dF(t).$$

Trigonometric series

- In a wider setting, this series may be the Fourier-Stieltjes series of a Borel measure μ, or, equivalently, of a function of bounded variation F (written $V_{\mathbb{T}}(F) < \infty$; we shall write just $V(F)$ if it is clear on which set the total variation is taken).
- In the former case the Fourier series is

$$f \sim \sum_{k=-\infty}^{\infty} \widehat{f}_k e^{ikt}, \qquad \widehat{f}_k = \frac{1}{2\pi} \int_{\mathbb{T}} f(t) e^{-ikt}\, dt,$$

- while in the latter case the Fourier-Stieltjes series is

$$dF \sim \sum_{k=-\infty}^{\infty} \widehat{dF}_k e^{ikt}, \qquad \widehat{dF}_k = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-ikt}\, dF(t).$$

- The latter case reduces to the former one if F is absolutely continuous with respect to the Lebesgue measure, whereas $dF(t) = f(t)\, dt$. The functions f and F are, in general, complex-valued and 2π-periodic.

Background

- During a more than one century, there were certain attempts to find necessary and sufficient conditions for the trigonometric series to be a Fourier series: Salem (1940), Edwards (1967), and others.

Background

- During a more than one century, there were certain attempts to find necessary and sufficient conditions for the trigonometric series to be a Fourier series: Salem (1940), Edwards (1967), and others.
- However, they were not given in terms of the coefficients of the initial trigonometric series nor were they practical enough.

Background

- During a more than one century, there were certain attempts to find necessary and sufficient conditions for the trigonometric series to be a Fourier series: Salem (1940), Edwards (1967), and others.
- However, they were not given in terms of the coefficients of the initial trigonometric series nor were they practical enough.
- Our main results are apparently the most general criterion, much more practical and is given by means of the coefficients as close as possible.

Background

- During a more than one century, there were certain attempts to find necessary and sufficient conditions for the trigonometric series to be a Fourier series: Salem (1940), Edwards (1967), and others.
- However, they were not given in terms of the coefficients of the initial trigonometric series nor were they practical enough.
- Our main results are apparently the most general criterion, much more practical and is given by means of the coefficients as close as possible.

- The difference between the latter and the formers seems to come out from the fact that while the earlier results only approached towards the Wiener algebra language,

Background

- During a more than one century, there were certain attempts to find necessary and sufficient conditions for the trigonometric series to be a Fourier series: Salem (1940), Edwards (1967), and others.
- However, they were not given in terms of the coefficients of the initial trigonometric series nor were they practical enough.
- Our main results are apparently the most general criterion, much more practical and is given by means of the coefficients as close as possible.

- The difference between the latter and the formers seems to come out from the fact that while the earlier results only approached towards the Wiener algebra language,
- our theorems exploit this setting explicitly and in its entirety.

Wiener algebras and trigonometric series

- One of our main results, not the most general, being in terms of a single, quite simple function (ℓ_c, piecewise linear, continuous and such that $\ell_c(k) = c_k, k \in \mathbb{Z}$) might be more practical in various applications.

Wiener algebras and trigonometric series

- One of our main results, not the most general, being in terms of a single, quite simple function (ℓ_c, piecewise linear, continuous and such that $\ell_c(k) = c_k, k \in \mathbb{Z}$) might be more practical in various applications.
- In the study of whether the trigonometric series is a Fourier series, our results amount to the following statement:

Criterion

For the series

$$\sum_{k=-\infty}^{\infty} c_k e^{ikt}$$

to be a Fourier series, it is necessary and sufficient that $\ell_c \in W_0(\mathbb{R})$.

- One of our main results, not the most general, being in terms of a single, quite simple function (ℓ_c, piecewise linear, continuous and such that $\ell_c(k) = c_k, k \in \mathbb{Z}$) might be more practical in various applications.
- In the study of whether the trigonometric series is a Fourier series, our results amount to the following statement:

Criterion

For the series

$$\sum_{k=-\infty}^{\infty} c_k e^{ikt}$$

to be a Fourier series, it is necessary and sufficient that $\ell_c \in W_0(\mathbb{R})$.

- Of course, this only gives a flavor of what is obtained and used, there are much more options, first of all for the algebras W or W^+ rather than just W_0. Goldberg

- The proven relations result in two ways (not like Robert Frost's two roads!) for obtaining new and known conditions for the trigonometric series to be a Fourier series or a Fourier-Stieltjes series.

Wiener algebras and trigonometric series

- The proven relations result in two ways (not like Robert Frost's two roads!) for obtaining new and known conditions for the trigonometric series to be a Fourier series or a Fourier-Stieltjes series.

- One of them is to search for specific conditions of belonging of the function ℓ_c to W_0, to W, or to another Wiener algebra. There is a chance for discoveries on this way but a general feeling is that in this case any activity will be equivalent to that with the sequence $\{c_k\}$. Just because of the structure of ℓ_c.

Wiener algebras and trigonometric series

- The proven relations result in two ways (not like Robert Frost's two roads!) for obtaining new and known conditions for the trigonometric series to be a Fourier series or a Fourier-Stieltjes series.
- One of them is to search for specific conditions of belonging of the function ℓ_c to W_0, to W, or to another Wiener algebra. There is a chance for discoveries on this way but a general feeling is that in this case any activity will be equivalent to that with the sequence $\{c_k\}$. Just because of the structure of ℓ_c.
- More promising seems the application of general conditions for belonging to W, W_0, etc. to the concrete function ℓ_c.

- Till recently, for deriving conditions to be a Fourier series, similar relations between trigonometric series and corresponding Fourier integrals has been known for functions of bounded variation.

Wiener algebras and trigonometric series

- Till recently, for deriving conditions to be a Fourier series, similar relations between trigonometric series and corresponding Fourier integrals has been known for functions of bounded variation.
- More precisely,

$$\sup_{|x|\le\pi}\left|\int_{\mathbb{R}} f(t)e^{ixt}dt-\sum_{k\in\mathbb{Z}} f(k)e^{ikx}\right|\le 2V_{\mathbb{R}}(f).$$

Wiener algebras and trigonometric series

- Till recently, for deriving conditions to be a Fourier series, similar relations between trigonometric series and corresponding Fourier integrals has been known for functions of bounded variation.
- More precisely,

$$\sup_{|x|\le\pi}\left|\int_{\mathbb{R}} f(t)e^{ixt}dt-\sum_{k\in\mathbb{Z}} f(k)e^{ikx}\right|\le 2V_{\mathbb{R}}(f).$$

- This result is due to Trigub (1980) (earlier version with f of compact support is due to Belinsky (1975); a general Euler-Maclaurin type formula is due to Trigub (1997)).

Wiener algebras and trigonometric series

- Till recently, for deriving conditions to be a Fourier series, similar relations between trigonometric series and corresponding Fourier integrals has been known for functions of bounded variation.
- More precisely,

$$\sup_{|x|\le\pi}\left|\int_{\mathbb{R}} f(t)e^{ixt}dt - \sum_{k\in\mathbb{Z}} f(k)e^{ikx}\right| \le 2V_{\mathbb{R}}(f).$$

- This result is due to Trigub (1980) (earlier version with f of compact support is due to Belinsky (1975); a general Euler-Maclaurin type formula is due to Trigub (1997)).
- Relating the problem to the Wiener algebras remove all the restrictions and allows one to reconfigure the process to functional analytic setting.

Wiener algebras and Fourier series: connections

- Roughly speaking, the trigonometric series is a Fourier-Stieltjes (Fourier) series if and only if the sequence of the coefficients can be extended on the whole real axis to a function from the corresponding Wiener algebra.

Wiener algebras and Fourier series: connections

- Roughly speaking, the trigonometric series is a Fourier-Stieltjes (Fourier) series if and only if the sequence of the coefficients can be extended on the whole real axis to a function from the corresponding Wiener algebra.
- We note that W reduces to W_0 in the case of absolutely continuous measures, and to the space of almost-periodic functions with absolutely convergent Fourier series if restricted to discrete measures.

Wiener algebras and Fourier series: connections

- Roughly speaking, the trigonometric series is a Fourier-Stieltjes (Fourier) series if and only if the sequence of the coefficients can be extended on the whole real axis to a function from the corresponding Wiener algebra.
- We note that W reduces to W_0 in the case of absolutely continuous measures, and to the space of almost-periodic functions with absolutely convergent Fourier series if restricted to discrete measures.
- If a function is in $W(\mathbb{R})$, it is uniformly continuous on $\mathbb{R}$, while a function $f \in W_0(\mathbb{R})$ also vanishes at infinity, $\lim\limits_{|t|\to\infty} f(t) = 0$, by the Riemann-Lebesgue lemma.

Wiener algebras and Fourier series: connections

- Roughly speaking, the trigonometric series is a Fourier-Stieltjes (Fourier) series if and only if the sequence of the coefficients can be extended on the whole real axis to a function from the corresponding Wiener algebra.
- We note that W reduces to W_0 in the case of absolutely continuous measures, and to the space of almost-periodic functions with absolutely convergent Fourier series if restricted to discrete measures.
- If a function is in $W(\mathbb{R})$, it is uniformly continuous on $\mathbb{R}$, while a function $f \in W_0(\mathbb{R})$ also vanishes at infinity, $\lim\limits_{|t|\to\infty} f(t) = 0$, by the Riemann-Lebesgue lemma.
- Outside of a neighborhood of infinity (understood as those t for which $|t| \ge M$) the structure of each of the three algebras is the same. Moreover, if a function is of bounded variation in a neighborhood of infinity and in W, and $f(\infty) = 0$, then it is in W_0.

Wiener algebras and Fourier series: connections

Theorem

In order that $\sum_{k=-\infty}^{\infty} c_k e^{ikt}$ *be the Fourier-Stieltjes series of* dF *(a Fourier series), it is necessary and sufficient that a function* $\phi \in W$ *(*$\phi \in W_0$*) exist such that* $\phi(k) = c_k$ *for all* $k \in \mathbb{Z}$*. By this,* $V(F) = \min_{\phi} \|\phi\|_W$*, where minimum is taken over all such* ϕ*. This minimum is attained at*

$$\phi_0(x) = \int_{\mathbb{T}} e^{-ixt}\, dF(t).$$

On the class of entire functions of exponential type not exceeding π*, all such extensions* ϕ *are of the form* $\phi(x) = \phi_0(x) + \lambda \sin \pi x$*, where* λ *is a number.*

The function F *is monotone increasing if and only if there exists a positive definite* ϕ*, that is,* $\phi \in W^+$*.*

Notations: ℓ_c and ℓ_f. Both are the piecewise linear continuous functions satisfying $\ell_c(k) = c_k$ and $\ell_f(k) = f(k)$, $k \in \mathbb{Z}$, respectively. More precisely, they are constructed by connecting the values at the integer points linearly. We shall also call these functions c-zigzag and f-zigzag, respectively.

Theorem

If $f \in W(\mathbb{R})$, then f-zigzag also belongs to $W(\mathbb{R})$, with $\|\ell_f\|_W \le \|f\|_W$. This inequality is sharp on the whole class. If $f \in W_0(\mathbb{R})$, or $f \in W_1(\mathbb{R})$, or $f \in W^+(\mathbb{R})$), then ℓ_f is such as well.

For positive definite functions (W^+), this theorem has long been known (Shepp; see Feller's book); many application in this case are found by Belov (2013). Many of the results of these two theorems can be found in a paper by Goldberg (1970), in a less precise form and without applications.

Wiener algebras and Fourier series: connections

- The following theorem is about how essential results for series can be derived from appropriate results for Wiener algebras.

Wiener algebras and Fourier series: connections

- The following theorem is about how essential results for series can be derived from appropriate results for Wiener algebras.
- We will say that a sequence $\{c_k\}$ is of bounded variation, written $\{c_k\} \in bv$, if
$$\sum_{k=-\infty}^{\infty} |c_k - c_{k+1}| < \infty.$$

Wiener algebras and Fourier series: connections

- The following theorem is about how essential results for series can be derived from appropriate results for Wiener algebras.
- We will say that a sequence $\{c_k\}$ is of bounded variation, written $\{c_k\} \in bv$, if
$$\sum_{k=-\infty}^{\infty} |c_k - c_{k+1}| < \infty.$$

Theorem

If $\sum_{k=-\infty}^{\infty} c_k e^{ikt}$ is a Fourier-Stieltjes series, the sequence $\{c_k\} \in bv$, and $\lim_{k\to\infty} [c_k - c_{-k}] = 0$, then

$$\sum_{k=-\infty}^{\infty} (c_k - \lim_{|k|\to\infty} c_k) e^{ikx}$$

is a Fourier series.

-

Connections between the Wiener algebras and Fourier series

The following theorem is of opposite nature: how a property of a Wiener algebra is derived from that for series.

Theorem

1) For each $f \in W_0(\mathbb{R})$, there exists a function g, with $g(|k|) \uparrow +\infty$ as $|k| \to \infty$, for which $gf \in W_0(\mathbb{R})$ as well.

2) If a sequence $\{\varepsilon_k\}$, $k \in \mathbb{N}$, is monotone decreasing to zero, then there is an even $f \in W_0^*(\mathbb{R})$ such that $f(k) \geq \varepsilon_k$, $k \in \mathbb{N}$.

Sufficient conditions

- The following assertion will give sufficient conditions for a function with compact support to belong to W_0.

Sufficient conditions

- The following assertion will give sufficient conditions for a function with compact support to belong to W_0.
- For this, recall certain notions.

Sufficient conditions

- The following assertion will give sufficient conditions for a function with compact support to belong to W_0.
- For this, recall certain notions.
- Let

$$\omega(f;h) = \sup_{0<\delta\le h} |f(x+\delta) - f(x)|$$

be the modulus of continuity of f in $C[-a,a]$, while

Sufficient conditions

- The following assertion will give sufficient conditions for a function with compact support to belong to W_0.
- For this, recall certain notions.
- Let

$$\omega(f;h) = \sup_{0<\delta\le h} |f(x+\delta) - f(x)|$$

be the modulus of continuity of f in $C[-a,a]$, while

- $$\omega(f;h)_2 = \sup_{0<\delta\le h} \left(\int_{-a}^{a} |f(x+\delta) - f(x)|^2 \, dx \right)^{\frac{1}{2}}$$

be the modulus of continuity in L_2.

Sufficient conditions

Theorem

If $f \in C[-a,a]$ *and* $\operatorname{supp} f \subset [-a+\varepsilon, a-\varepsilon]$, *for some* $\varepsilon > 0$, *then for* $f \in W_0$ *it suffices, and if* $|\widehat{f}(\frac{k\pi}{a})|$ *is monotone decreasing as* $|k| \to \infty$ *is also necessary, that*

$$\int_0^1 \frac{\omega(f;t)_2}{\sqrt{t}}\, dt < \infty.$$

If also $V_{[-a,a]}(f) < \infty$, *the sufficient condition is*

$$\int_0^1 \frac{\sqrt{\omega(f;t)}}{t}\, dt < \infty.$$

The latter condition is sharp on the considered class.

We note that if $V_{[-a,a]}(f) < \infty$, it suffices, for $h \to +0$, that $\omega(f;h) = O\big(\frac{1}{\ln^{2+\varepsilon} \frac{1}{h}}\big)$.

Completely new sufficient conditions

If $\lim\limits_{|k|\to\infty} c_k = 0$, the sequence $\{c_k\}$ is called a null-sequence.

Theorem

Let the sequence of the coefficients of $\sum\limits_{k=-\infty}^{\infty} c_k e^{ikt}$ *be a null-sequence.*

1) If $\sum\limits_{m=1}^{\infty} \frac{1}{m} \left(\sum\limits_{n=m}^{\infty} \sup\limits_{|k|\ge n} |c_k| \sup\limits_{|k|\ge n} |c_k - c_{k+1}| \right)^{\frac{1}{2}} < \infty$, *then this series is a Fourier series.*

2) If there is $p \in (0, 2]$ *such that* $\{c_k\} \in l_p$, *then this series is the Fourier series of an* L_2 *function.*
If $\{c_k\} \in l_p$ *only for some* $p \in (2, +\infty)$, *then assuming* $\{c_k - c_{k+1}\} \in l_q$, *with* $q \in \left(0, 1 + \frac{1}{p-1}\right)$, *we get that this series is a Fourier series.*

Completely new sufficient conditions

- In both cases we apply to ℓ_c earlier (but recent) results for belonging to $W_0(\mathbb{R})$.

Completely new sufficient conditions

- In both cases we apply to ℓ_c earlier (but recent) results for belonging to $W_0(\mathbb{R})$.
- 1) is derived from a result due to L-Trigub (2011): $f \in W_0(\mathbb{R})$ provided that f is locally absolutely continuous on $\mathbb{R}$, $f(x) = 0$ if $|x| \le 1$, $\lim\limits_{|x|\to\infty} f(x) = 0$, and

$$\int_1^\infty \left(\int_t^\infty \sup_{|u|\ge y\ge 1} |f(u)| \operatorname*{esssup}_{|u|\ge y\ge 1} |f'(u)|\, dy \right)^{\frac{1}{2}} \frac{dt}{t} < \infty.$$

Completely new sufficient conditions

- In both cases we apply to ℓ_c earlier (but recent) results for belonging to $W_0(\mathbb{R})$.
- 1) is derived from a result due to L-Trigub (2011): $f \in W_0(\mathbb{R})$ provided that f is locally absolutely continuous on $\mathbb{R}$, $f(x) = 0$ if $|x| \le 1$, $\lim\limits_{|x|\to\infty} f(x) = 0$, and

$$\int_1^\infty \left(\int_t^\infty \sup_{|u|\ge y\ge 1} |f(u)| \operatorname*{esssup}_{|u|\ge y\ge 1} |f'(u)|\, dy \right)^{\frac{1}{2}} \frac{dt}{t} < \infty.$$

- 2) follows from a result due to L (2011): If f is in L_p, $1 \le p < \infty$, and its derivative is in L_q, $1 < q < \infty$, then the function is in $W_0(\mathbb{R})$ provided that $\frac{1}{p} + \frac{1}{q} > 1$.

Completely new sufficient conditions

- In both cases we apply to ℓ_c earlier (but recent) results for belonging to $W_0(\mathbb{R})$.
- 1) is derived from a result due to L-Trigub (2011): $f \in W_0(\mathbb{R})$ provided that f is locally absolutely continuous on $\mathbb{R}$, $f(x) = 0$ if $|x| \le 1$, $\lim\limits_{|x|\to\infty} f(x) = 0$, and

$$\int_1^\infty \left(\int_t^\infty \sup_{|u|\ge y\ge 1} |f(u)| \operatorname*{esssup}_{|u|\ge y\ge 1} |f'(u)|\, dy \right)^{\frac{1}{2}} \frac{dt}{t} < \infty.$$

- 2) follows from a result due to L (2011): If f is in L_p, $1 \le p < \infty$, and its derivative is in L_q, $1 < q < \infty$, then the function is in $W_0(\mathbb{R})$ provided that $\frac{1}{p} + \frac{1}{q} > 1$.
- Transferring as above the well known tests for belonging to W_0 due to Titchmarsh, Beurling or Carleman to Fourier series gives nothing beyond what we already know.

Hilbert transforms

- The following results are somewhat more restrictive, since they use the traditional assumption for the sequence to be of bounded variation.

Hilbert transforms

- The following results are somewhat more restrictive, since they use the traditional assumption for the sequence to be of bounded variation.
- In the preceding theorem there are no a priori claims, except the natural one that the terms tend to zero.

Hilbert transforms

- The following results are somewhat more restrictive, since they use the traditional assumption for the sequence to be of bounded variation.
- In the preceding theorem there are no a priori claims, except the natural one that the terms tend to zero.
- For a (complex-valued) function $f \in L_1(\mathbb{R})$, its Hilbert transform $\mathcal{H}f$ is defined by

$$\mathcal{H}f(x) := \frac{1}{\pi}\,(\text{P.V.}) \int_{\mathbb{R}} \frac{f(u)}{x-u}\,du, \quad x \in \mathbb{R}.$$

Hilbert transforms

- The following results are somewhat more restrictive, since they use the traditional assumption for the sequence to be of bounded variation.
- In the preceding theorem there are no a priori claims, except the natural one that the terms tend to zero.
- For a (complex-valued) function $f \in L_1(\mathbb{R})$, its Hilbert transform $\mathcal{H}f$ is defined by

$$\mathcal{H}f(x) := \frac{1}{\pi}\,(\mathrm{P.V.}) \int_{\mathbb{R}} \frac{f(u)}{x-u}\,du, \quad x \in \mathbb{R}.$$

- As is well known, for $f \in L_1(\mathbb{R})$, this limit exists for almost all $x \in \mathbb{R}$ but is not necessarily integrable. However, we always have $\|\mathcal{H}f\|_W = \|f\|_W$.

Hilbert transforms

- The following results are somewhat more restrictive, since they use the traditional assumption for the sequence to be of bounded variation.
- In the preceding theorem there are no a priori claims, except the natural one that the terms tend to zero.
- For a (complex-valued) function $f \in L_1(\mathbb{R})$, its Hilbert transform $\mathcal{H}f$ is defined by

$$\mathcal{H}f(x) := \frac{1}{\pi}\,(\text{P.V.})\int_{\mathbb{R}} \frac{f(u)}{x-u}\,du, \quad x \in \mathbb{R}.$$

- As is well known, for $f \in L_1(\mathbb{R})$, this limit exists for almost all $x \in \mathbb{R}$ but is not necessarily integrable. However, we always have $\|\mathcal{H}f\|_W = \|f\|_W$.
- Analogously, for a sequence $c = \{c_k\}$, its discrete Hilbert transform hc can be defined (there are other equivalent definitions) as the sequence $\{hc_n\}$: $\qquad hc_n = \sum\limits_{k=-\infty}^{\infty} \frac{c_k}{n+\frac{1}{2}-k}.$

Sufficient conditions

Theorem

Let the sequence of the coefficients of $\sum_{k=-\infty}^{\infty} c_k e^{ikt}$ *be a* bv *null-sequence.*
Each of the following additional conditions:
1) the discrete Hilbert transform of the sequence $d = \{d_k\} := \{c_{k+1} - c_k\}$ *is summable, written* $hd \in l_1$,
and
2) the discrete Hilbert transform $hc \in bv$,
guarantees for $\sum_{k=-\infty}^{\infty} c_k e^{ikt}$ *to be a Fourier series.*

Item 2) is a kind of an analog of the following Hardy-Littlewood theorem:
If a periodic function and its conjugate are both of bounded variation, then their Fourier series converge absolutely.

Necessary conditions

We begin with a general one.

Lemma

If $f \in W$ *is represented by*

$$f(x) = \int_{\mathbb{R}} e^{-ixt}\, dF(t),$$

then, for every $x \in \mathbb{R}$*, the improper integral*

$$\int_{\to+0}^{\to+\infty} \frac{f(x+u) - f(x-u)}{u}\, du = -\pi i \int_{\mathbb{R}} e^{-ixt} \operatorname{sign} t\, dF(t)$$

converges, and

$$\sup_{x,\delta,M} \left| \int_{\delta}^{M} \frac{f(x+u) - f(x-u)}{u}\, du \right| < \infty.$$

The improper integral may not converge absolutely.

Necessary conditions

- We mention that on the intervals of monotonicity of f, when the difference $f(x+u)-f(x-u)$ is also monotone in u, the convergence of the indicated integral yields $o(\frac{1}{\ln\frac{1}{u}})$ for this difference, as $u\to+0$.

Necessary conditions

- We mention that on the intervals of monotonicity of f, when the difference $f(x+u)-f(x-u)$ is also monotone in u, the convergence of the indicated integral yields $o(\frac{1}{\ln\frac{1}{u}})$ for this difference, as $u\to+0$.
- This easily implies the existence of an even trigonometric series with the coefficients, monotonously tending to zero, which is not a Fourier series (the first such example was constructed by Sidon (1921)) as well as the existence of an even function tending to zero which does not belong to $W(\mathbb{R})$.

Necessary conditions

- We mention that on the intervals of monotonicity of f, when the difference $f(x+u)-f(x-u)$ is also monotone in u, the convergence of the indicated integral yields $o(\frac{1}{\ln\frac{1}{u}})$ for this difference, as $u \to +0$.
- This easily implies the existence of an even trigonometric series with the coefficients, monotonously tending to zero, which is not a Fourier series (the first such example was constructed by Sidon (1921)) as well as the existence of an even function tending to zero which does not belong to $W(\mathbb{R})$.
- We observe that if f is odd, continuous and monotone near infinity, the integral $\int_{\to 0}^{\to\infty} \frac{f(x)}{x}\,dx$ converges. This condition is also sufficient. To prove this, the similar condition for the sine Fourier series and the first main theorem should be used.

Necessary conditions

- We mention that on the intervals of monotonicity of f, when the difference $f(x+u) - f(x-u)$ is also monotone in u, the convergence of the indicated integral yields $o(\frac{1}{\ln \frac{1}{u}})$ for this difference, as $u \to +0$.
- This easily implies the existence of an even trigonometric series with the coefficients, monotonously tending to zero, which is not a Fourier series (the first such example was constructed by Sidon (1921)) as well as the existence of an even function tending to zero which does not belong to $W(\mathbb{R})$.
- We observe that if f is odd, continuous and monotone near infinity, the integral $\int_{\to 0}^{\to\infty} \frac{f(x)}{x}\,dx$ converges. This condition is also sufficient. To prove this, the similar condition for the sine Fourier series and the first main theorem should be used.
- In particular, $f(x) = o\big(\frac{1}{\ln |x|}\big)$ as $|x| \to \infty$.

Necessary conditions

- We mention that on the intervals of monotonicity of f, when the difference $f(x+u)-f(x-u)$ is also monotone in u, the convergence of the indicated integral yields $o(\frac{1}{\ln \frac{1}{u}})$ for this difference, as $u \to +0$.
- This easily implies the existence of an even trigonometric series with the coefficients, monotonously tending to zero, which is not a Fourier series (the first such example was constructed by Sidon (1921)) as well as the existence of an even function tending to zero which does not belong to $W(\mathbb{R})$.
- We observe that if f is odd, continuous and monotone near infinity, the integral $\int_{\to 0}^{\to\infty} \frac{f(x)}{x}\,dx$ converges. This condition is also sufficient. To prove this, the similar condition for the sine Fourier series and the first main theorem should be used.
- In particular, $f(x) = o\big(\frac{1}{\ln|x|}\big)$ as $|x| \to \infty$.
- The answer is different for even functions from W_0, which can decay arbitrarily slowly. In this case $f(x-1)-f(x+1) = O\left(\frac{1}{\ln x}\right)$ as $x \to +\infty$.

Necessary conditions

- In Chapter II, §9 of Bary's book: Salem's necessary conditions for $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ to be a Fourier series (1940).

Necessary conditions

- In Chapter II, §9 of Bary's book: Salem's necessary conditions for $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ to be a Fourier series (1940).
- Salem's results say that necessary conditions for a_n and b_n to be the cosine Fourier coefficients and the sine Fourier coefficients, are

$$\lim_{k\to\infty} k \sum_{n=1}^{\infty} \frac{a_n}{(k+\frac{1}{2})^2 - n^2} = 0, \tag{1}$$

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} \frac{nb_n}{(k+\frac{1}{2})^2 - n^2} = 0. \tag{2}$$

Necessary conditions

- In Chapter II, §9 of Bary's book: Salem's necessary conditions for $\frac{a_0}{2} + \sum\limits_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ to be a Fourier series (1940).
- Salem's results say that necessary conditions for a_n and b_n to be the cosine Fourier coefficients and the sine Fourier coefficients, are

$$\lim_{k\to\infty} k \sum_{n=1}^{\infty} \frac{a_n}{(k+\frac{1}{2})^2 - n^2} = 0, \tag{1}$$

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} \frac{nb_n}{(k+\frac{1}{2})^2 - n^2} = 0. \tag{2}$$

- A today glance at these relations allows one to immediately recognize in the expressions under the limit sign in (1) and (2) the discrete even and odd Hilbert transforms, respectively.

Necessary conditions

- In Chapter II, §9 of Bary's book: Salem's necessary conditions for $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ to be a Fourier series (1940).
- Salem's results say that necessary conditions for a_n and b_n to be the cosine Fourier coefficients and the sine Fourier coefficients, are

$$\lim_{k\to\infty} k \sum_{n=1}^{\infty} \frac{a_n}{(k+\frac{1}{2})^2 - n^2} = 0, \tag{1}$$

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} \frac{nb_n}{(k+\frac{1}{2})^2 - n^2} = 0. \tag{2}$$

- A today glance at these relations allows one to immediately recognize in the expressions under the limit sign in (1) and (2) the discrete even and odd Hilbert transforms, respectively.
- Of course, it was not the case in time of Salem's publication nor of Bary's.

Necessary conditions

- The following arguments give enriched perception of these conditions.

Necessary conditions

- The following arguments give enriched perception of these conditions.
- If in the odd case the coefficients b_k decrease monotone, then, $\sum_{n=1}^{\infty} b_n \sin nx$ is a Fourier series if and only if $\sum \frac{b_k}{k} < \infty$.

Necessary conditions

- The following arguments give enriched perception of these conditions.
- If in the odd case the coefficients b_k decrease monotone, then, $\sum_{n=1}^{\infty} b_n \sin nx$ is a Fourier series if and only if $\sum \frac{b_k}{k} < \infty$.
- A different matter is the even case: the coefficients may decay arbitrarily slow.

Necessary conditions

- The following arguments give enriched perception of these conditions.
- If in the odd case the coefficients b_k decrease monotone, then, $\sum_{n=1}^{\infty} b_n \sin nx$ is a Fourier series if and only if $\sum \frac{b_k}{k} < \infty$.
- A different matter is the even case: the coefficients may decay arbitrarily slow.
- Taking into account a possibility of relation between the Fourier expansion and Hilbert transform, a necessary condition for a function to belong to Wiener's algebra has been obtained. It reads as follows: *If* $f \in W_0(\mathbb{R})$, *then*

$$\lim_{|x|\to\infty} \mathcal{H}f(x) = 0.$$

Necessary conditions

- The following arguments give enriched perception of these conditions.
- If in the odd case the coefficients b_k decrease monotone, then, $\sum_{n=1}^{\infty} b_n \sin nx$ is a Fourier series if and only if $\sum \frac{b_k}{k} < \infty$.
- A different matter is the even case: the coefficients may decay arbitrarily slow.
- Taking into account a possibility of relation between the Fourier expansion and Hilbert transform, a necessary condition for a function to belong to Wiener's algebra has been obtained. It reads as follows: *If* $f \in W_0(\mathbb{R})$, *then*

$$\lim_{|x|\to\infty} \mathcal{H}f(x) = 0.$$

- For $f \in W_0(\mathbb{R})$, its Hilbert transform exists everywhere.

Necessary conditions

- We can derive Salem's conditions in the following form.

Necessary conditions

- We can derive Salem's conditions in the following form.
- If $\sum_{k=-\infty}^{\infty} c_k e^{ikt}$ is a Fourier series, then the discrete Hilbert transform hc_n exists for all $n = 0, \pm 1, \pm 2, ...$, and

$$\lim_{|n|\to\infty} hc_n = \lim_{|n|\to\infty} \sum_{k=-\infty}^{\infty} \frac{c_k}{n + \frac{1}{2} - k} = 0. \tag{3}$$

Necessary conditions

- We can derive Salem's conditions in the following form.
- If $\sum_{k=-\infty}^{\infty} c_k e^{ikt}$ is a Fourier series, then the discrete Hilbert transform hc_n exists for all $n = 0, \pm 1, \pm 2, ...$, and

$$\lim_{|n|\to\infty} hc_n = \lim_{|n|\to\infty} \sum_{k=-\infty}^{\infty} \frac{c_k}{n + \frac{1}{2} - k} = 0. \tag{3}$$

- In words, the necessary condition for the trigonometric series to be a Fourier series is of the same form as that for a function to be in the Wiener algebra: the Hilbert transform vanishes at infinity. Of course, in the case of series the transform is discrete.

Necessary conditions

- We can derive Salem's conditions in the following form.
- If $\sum_{k=-\infty}^{\infty} c_k e^{ikt}$ is a Fourier series, then the discrete Hilbert transform hc_n exists for all $n = 0, \pm 1, \pm 2, ...$, and

$$\lim_{|n|\to\infty} hc_n = \lim_{|n|\to\infty} \sum_{k=-\infty}^{\infty} \frac{c_k}{n + \frac{1}{2} - k} = 0. \qquad (3)$$

- In words, the necessary condition for the trigonometric series to be a Fourier series is of the same form as that for a function to be in the Wiener algebra: the Hilbert transform vanishes at infinity. Of course, in the case of series the transform is discrete.
- If one decides to begin with $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, then (3) easily reduces to (1) and (2).

Necessary conditions

- Similarly, in the case of f even or odd, the condition $\lim_{|x|\to\infty} \mathcal{H}f(x) = 0$ reduces to

Necessary conditions

- Similarly, in the case of f even or odd, the condition $\lim_{|x|\to\infty} \mathcal{H}f(x) = 0$ reduces to
- $$\lim_{x\to\infty} x \int_0^\infty \frac{f(t)}{x^2 - t^2}\, dt = 0$$

Necessary conditions

- Similarly, in the case of f even or odd, the condition $\lim_{|x|\to\infty} \mathcal{H}f(x) = 0$ reduces to
- $$\lim_{x\to\infty} x \int_0^\infty \frac{f(t)}{x^2 - t^2}\, dt = 0$$
- and
$$\lim_{x\to\infty} \int_0^\infty \frac{t f(t)}{x^2 - t^2}\, dt = 0,$$

  respectively.

Necessary conditions

- Similarly, in the case of f even or odd, the condition $\lim\limits_{|x|\to\infty} \mathcal{H}f(x) = 0$ reduces to
- $$\lim_{x\to\infty} x \int_0^\infty \frac{f(t)}{x^2 - t^2}\, dt = 0$$
- and
$$\lim_{x\to\infty} \int_0^\infty \frac{tf(t)}{x^2 - t^2}\, dt = 0,$$
respectively.
- Multidimensional case: Trigub

Necessary conditions

- Similarly, in the case of f even or odd, the condition $\lim\limits_{|x|\to\infty} \mathcal{H}f(x) = 0$ reduces to
- $$\lim_{x\to\infty} x \int_0^\infty \frac{f(t)}{x^2 - t^2}\, dt = 0$$
- and
$$\lim_{x\to\infty} \int_0^\infty \frac{tf(t)}{x^2 - t^2}\, dt = 0,$$
respectively.
- Multidimensional case: Trigub
- THANK YOU!