Subordination principle for the space-time-fractional diffusion equations

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Outline of the talk:

- Earlier results on subordination principles
- Space-time-fractional diffusion equation and its fundamental solution
- Subordination principle for the space-time-fractional diffusion equations
- A new class of probability density functions
- Short survey of other related results

Weierstrass formula

Let $S_1(t)x := u(t)$ be the solution (semi-group) operator to the abstract Cauchy problem

$$u'(t) = Au(t),\ t > 0,\ u(0) = x,$$

and $S_2(t)x := u(t)$ be the solution operator (cosine family) to the abstract Cauchy problem

$$u''(t) = Au(t),\ t > 0,\ u(0) = x,\ u'(0) = 0.$$

Then the abstract Weierstrass formula is valid:

$$S_1(t)x = \int_0^{+\infty} \frac{e^{-\tau^2/(4t)}}{\sqrt{\pi t}} S_2(\tau)x\, d\tau,\ t > 0.$$

General results

A general subordination principle for completely positive measures was introduced in

J. Prüss, *Evolutionary Integral Equations and Applications*, Birkhäuser, Basel, 1993

and applied for constructing new resolvents for the abstract Volterra integral equations based on the known ones.

Subordination principle for the fractional abstract evolution equation

Extension and specialization of the Prüss subordination principle for the abstract fractional evolution equation

$$D^{\beta}u(t) = Au(t),\ 0 < \beta \leq 2$$

subject to the initial condition $(0 < \beta \leq 1)$

$$u(0) = x,$$

or to the initial conditions $(1 < \beta \leq 2)$

$$u(0) = x,\ u'(0) = 0,$$

where D^{β} is the Dzherbashyan-Caputo fractional derivative of order $\beta,\ 0 < \beta \leq 2$ and A is a linear closed unbounded operator densely defined in a Banach space $X,\ x \in X$:
E. Bajlekova, *Fractional Evolution Equations in Banach Spaces*, Ph.D. thesis, University of Eindhoven, The Netherlands, 2001.

The Dzherbashyan-Caputo fractional derivative

The Dzherbashyan-Caputo fractional derivative of order $\beta,\ n-1 < \beta \le n,\ n \in \mathbf{N}$:

$$(D^{\beta} f)(x) = (I^{n-\alpha} f^{(n)})(x)$$

with the Riemann-Liouville fractional integral $I^{n-\alpha}$ defined by the formula

$$(I^{\alpha} f)(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt, & \alpha > 0, \\ f(x), & \alpha = 0. \end{cases}$$

For the Dzherbashyan-Caputo fractional derivative, the 1st Fundamental Theorem of FC is valid:

$$(D^{\alpha} I^{\alpha} f)(x) = f(x).$$

Subordination principle for the fractional abstract evolution equation

Let $0 < \beta < \delta \le 2$, $\gamma = \beta/\delta$ and let $S_\beta(t)x$ be a solution operator for

$$D^\beta u(t) = Au(t),\ 0 < \beta \le 2$$

subject to the initial condition $u(0) = x$ $(0 < \beta \le 1)$ or to the initial conditions $u(0) = x,\ u'(0) = 0$ $(1 < \beta \le 2)$.
Then the subordination formula

$$S_\beta(t)x = \int_0^\infty t^{-\gamma} W_{1-\gamma,-\gamma}(-\tau t^{-\gamma}) S_\delta(\tau)x\, d\tau,\ t > 0,\ x \in X$$

is valid under some conditions on the operator A. $W_{1-\gamma,-\gamma}$ is a special case of the Wright function

$$W_{a,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(a+\mu k)},\ \mu > -1,\ a,\, z \in \mathbb{C}.$$

The kernel function $t^{-\gamma} W_{1-\gamma,-\gamma}(-\tau t^{-\gamma})$ can be interpreted as a unilateral probability density function in τ evolving in time $(t > 0)$.

Subordination principles for the one-dimensional diffusion-wave equation

The case of the space-time-fractional diffusion-wave equation

$$_tD^{\beta}\,u(x,t) \;=\; {_xD^{\alpha}_{\theta}}\,u(x,t)\,,\;\; x\in \boldsymbol{R}\,,\;\; t\in \boldsymbol{R}^{+}\,,$$

where $0<\alpha\leq 2\,,\;|\theta|\leq \min\{\alpha, 2-\alpha\}\,,\;0<\beta\leq 2\,,\;{_tD^{\beta}}$ is the Dzherbashyan-Caputo time-fractional derivative of order β and ${_xD^{\alpha}_{\theta}}$ is the *Riesz-Feller* space-fractional derivative of order α and skewness θ:

$$(\mathcal{F}\,{_xD^{\alpha}_{\theta}}\,f)\,(\kappa) = -|\kappa|^{\alpha}\,\mathrm{e}^{i(\mathrm{sign}\,\kappa)\theta\pi/2}\,(\mathcal{F}\,f)(\kappa)$$

was treated in
F. Mainardi, Yu. Luchko, G. Pagnini, *The fundamental solution of the space-time fractional diffusion equation*, Fract. Calc. Appl. Anal. **4** (2001), 153–192

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Multi-dimensional space-time-fractional diffusion-wave equations

Today we treat the following Cauchy problem:

$$D_t^{\beta} u(\mathrm{x}, t) = -(-\Delta)^{\frac{\alpha}{2}} u(\mathrm{x}, t), \quad \mathrm{x} \in \mathbb{R}^n,\ t > 0,\ 0 < \alpha \leqslant 2,\ 0 < \beta \leqslant 2,$$

where D_t^{β} is the Dzherbashyan-Caputo time-fractional derivative of the order β and $-(-\Delta)^{\frac{\alpha}{2}}$ is the fractional Laplace operator (Riesz space-fractional derivative of the order $\alpha,\ \alpha > 0$):

$$\left(\mathcal{F} - (-\Delta)^{\frac{\alpha}{2}} f\right)(\kappa) = -|\kappa|^{\alpha}(\mathcal{F} f)(\kappa)$$

along with the initial conditions $(0 < \beta \leq 1)$

$$u(\mathrm{x}, 0) = \varphi(\mathrm{x}), \quad \mathrm{x} \in \mathbb{R}^n,$$

or $(1 < \beta \leq 2)$

$$u(\mathrm{x}, 0) = \varphi(\mathrm{x}),\ \frac{\partial u}{\partial t}(\mathrm{x}, 0) = 0, \quad \mathrm{x} \in \mathbb{R}^n.$$

Fractional Laplace operator

For $0 < \alpha < m$, $m \in \mathbb{N}$ and $x \in \mathbb{R}^n$, the fractional Laplace operator (the Riesz space-fractional derivative) can be also represented as a hypersingular integral:

$$-(-\Delta)^{\frac{\alpha}{2}} f(\mathrm{x}) = -\frac{1}{d_{n,m}(\alpha)} \int_{\mathbb{R}^n} \frac{(\Delta_{\mathrm{h}}^m f)(\mathrm{x})}{|\mathrm{h}|^{n+\alpha}} \, d\mathrm{h}$$

with the suitably defined finite differences operator $(\Delta_{\mathrm{h}}^m f)(\mathrm{x})$ and the normalization constant $d_{n,m}(\alpha)$.

M. Kwasnicki, TEN EQUIVALENT DEFINITIONS OF THE FRACTIONAL LAPLACE OPERATOR. Fract. Calc. Appl. Anal., Vol. 20, No 1 (2017), pp. 7-51.

Fundamental solution $G_{\alpha,\beta,n}$

Because the problem is a linear one, its solution can be represented in the form

$$u(\mathrm{x}, t) = \int_{\mathbb{R}^n} G_{\alpha,\beta,n}(\mathrm{x} - \zeta, t)\varphi(\zeta)\, d\zeta,$$

where $G_{\alpha,\beta,n}$ is the first fundamental solution, i.e., the solution to the Cauchy problem with the initial conditions

$$u(\mathrm{x}, 0) = \prod_{i=1}^{n} \delta(x_i)\,, \quad \mathrm{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$$

or

$$u(\mathrm{x}, 0) = \prod_{i=1}^{n} \delta(x_i)\,, \quad \mathrm{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$$

and

$$\frac{\partial u}{\partial t}(\mathrm{x}, 0) = 0\,, \quad \mathrm{x} \in \mathbb{R}^n,$$

for $0 < \beta \leqslant 1$ or $1 < \beta \leqslant 2$, respectively, with δ being the Dirac delta function.

Fundamental solution $G_{\alpha,\beta,n}$

I) Application of the multi-dimensional Fourier transform:

$$D_t^{\beta}\hat{G}_{\alpha,\beta,n}(\kappa,t) + |\kappa|^{\alpha}\hat{G}_{\alpha,\beta,n}(\kappa,t) = 0,$$

along with the initial conditions

$$\hat{G}_{\alpha,\beta,n}(\kappa,0) = 1$$

in the case $0 < \beta \leq 1$ or with the initial conditions

$$\hat{G}_{\alpha,\beta,n}(\kappa,0) = 1,\ \frac{\partial}{\partial t}\hat{G}_{\alpha,\beta,n}(\kappa,0) = 0$$

in the case $1 < \beta \leq 2$.

Fundamental solution $G_{\alpha,\beta,n}$

II) In both cases, the unique solution has the following form:

$$\hat{G}_{\alpha,\beta,n}(\kappa, t) = E_\beta\left(-|\kappa|^\alpha t^\beta\right)$$

in terms of the Mittag-Leffler function $E_\beta(z)$ that is defined by a convergent series

$$E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+\beta\, n)}\,, \quad \beta > 0,\ z \in \mathbb{C}.$$

III) Application of the inverse Fourier transform:

$$G_{\alpha,\beta,n}(\mathrm{x}, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\kappa\cdot\mathrm{x}} E_\beta\left(-|\kappa|^\alpha t^\beta\right)\, d\kappa\,, \quad \mathrm{x} \in \mathbb{R}^n\,, t > 0\,.$$

Fundamental solution $G_{\alpha,\beta,n}$

IV) $E_\beta\left(-|\kappa|^\alpha t^\beta\right)$ is a radial function and thus the formula

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i\kappa\cdot \mathrm{x}}\varphi(|\kappa|)\,d\kappa = \frac{|\mathrm{x}|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}}\int_0^\infty \varphi(\tau)\tau^{\frac{n}{2}}J_{\frac{n}{2}-1}(\tau|\mathrm{x}|)\,d\tau$$

leads to the representation

$$G_{\alpha,\beta,n}(\mathrm{x},t) = \frac{|\mathrm{x}|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}}\int_0^\infty E_\beta\left(-\tau^\alpha t^\beta\right)\tau^{\frac{n}{2}}J_{\frac{n}{2}-1}(\tau|\mathrm{x}|)\,d\tau,$$

where $J_{\frac{n}{2}-1}$ is the Bessel function with the index $\frac{n}{2}-1$ and E_β is the Mittag-Leffler function.

Yu. Luchko, *Multi-dimensional fractional wave equation and some properties of its fundamental solution*, Communications in Applied and Industrial Mathematics **6** (2014), e-485.

Mellin convolution theorem

The Mellin integral transform of a function $f = f(t),\ t > 0$ at the point $s \in \mathcal{C}$:

$$\{\mathcal{M}\, f;\, s\} = f^*(s) = \int_0^{+\infty} f(t)\, t^{s-1}\, dt$$

The inverse Mellin integral transform:

$$f(t) = (\mathcal{M}^{-1} f^*(s))(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s) t^{-s}\, ds\,, \gamma_1 < \Re(s) = \gamma < \gamma_2\,.$$

If we denote by $\stackrel{\mathcal{M}}{\longleftrightarrow}$ the juxtaposition of a function f with its Mellin transform f^* then the convolution theorem for the Mellin integral transform reads as

$$\int_0^{\infty} f_1(\tau) f_2\left(\frac{y}{\tau}\right) \frac{d\tau}{\tau} \stackrel{\mathcal{M}}{\longleftrightarrow} f_1^*(s) f_2^*(s)\,.$$

Mellin convolution theorem

For $\mathrm{x} \neq 0$ the integral

$$G_{\alpha,\beta,n}(\mathrm{x},t) = \frac{|\mathrm{x}|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}} \int_0^\infty E_\beta\left(-\tau^\alpha t^\beta\right) \tau^{\frac{n}{2}} J_{\frac{n}{2}-1}(\tau|\mathrm{x}|)\, d\tau$$

can be interpreted as the Mellin convolution of the functions

$$f_1(\tau) = E_\beta(-\tau^\alpha\, t^\beta) \quad \text{and} \quad f_2(\tau) = \frac{|\mathrm{x}|^{-n}}{(2\pi)^{\frac{n}{2}}}\, \tau^{-\frac{n}{2}-1}\, J_{\frac{n}{2}-1}\left(\frac{1}{\tau}\right)$$

evaluated at the point $y = \frac{1}{|\mathrm{x}|}$.

Mellin-Barnes representations of $G_{\alpha,\beta,n}$

The formulas

$$f_1^*(s) = \frac{t^{-\frac{\beta}{\alpha}s}}{\alpha} \frac{\Gamma(\frac{s}{\alpha})\Gamma(1-\frac{s}{\alpha})}{\Gamma(1-\frac{\beta}{\alpha}s)}$$

$$f_2^*(s) = \frac{|\mathrm{x}|^{-n}}{(2\pi)^{\frac{n}{2}}} \left(\frac{1}{2}\right)^{-\frac{n}{2}+s} \frac{\Gamma\left(\frac{n}{2}-\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}$$

along with the convolution theorem and inverse Mellin transform lead to the following Mellin-Barnes integral representation of the fundamental solution $G_{\alpha,\beta,n}$:

$$G_{\alpha,\beta,n}(\mathrm{x},t) = \frac{1}{\alpha}\frac{|\mathrm{x}|^{-n}}{\pi^{\frac{n}{2}}}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma\left(\frac{n}{2}-\frac{s}{2}\right)\Gamma\left(\frac{s}{\alpha}\right)\Gamma\left(1-\frac{s}{\alpha}\right)}{\Gamma\left(1-\frac{\beta}{\alpha}s\right)\Gamma\left(\frac{s}{2}\right)} \left(\frac{2t^{\frac{\beta}{\alpha}}}{|\mathrm{x}|}\right)^{-s} ds.$$

Mellin-Barnes representations of $G_{\alpha,\beta,n}$

By some simple linear variables substitutions we get the representation

$$G_{\alpha,\beta,n}(\mathrm{x},t) = \frac{1}{\alpha}\frac{t^{-\frac{\beta n}{\alpha}}}{(4\pi)^{\frac{n}{2}}}\frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{n}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(1-\frac{n}{\alpha}+\frac{s}{\alpha}\right)}{\Gamma\left(1-\frac{\beta}{\alpha}n+\frac{\beta}{\alpha}s\right)\Gamma\left(\frac{n}{2}-\frac{s}{2}\right)}\left(\frac{|\mathrm{x}|}{2t^{\frac{\beta}{\alpha}}}\right)^{-s}ds$$

that can be rewritten in the form

$$G_{\alpha,\beta,n}(\mathrm{x},t) = \frac{1}{\alpha}\frac{t^{-\frac{\beta n}{\alpha}}}{(4\pi)^{\frac{n}{2}}}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}K_{\alpha,\beta,n}(s)z^{-s}ds\,, \quad z = \frac{|\mathrm{x}|}{2t^{\frac{\beta}{\alpha}}}$$

with

$$K_{\alpha,\beta,n}(s) = \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{n}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(1-\frac{n}{\alpha}+\frac{s}{\alpha}\right)}{\Gamma\left(1-\frac{\beta}{\alpha}n+\frac{\beta}{\alpha}s\right)\Gamma\left(\frac{n}{2}-\frac{s}{2}\right)}.$$

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General subordination principle

Let

$$G_{\alpha,\beta,n}(\mathrm{x},t) = \int_0^\infty \Phi(\tau,t)\, G_{\hat\alpha,\hat\beta,n}(\mathrm{x},\tau)\, d\tau,$$

where the kernel function $\Phi = \Phi(\tau, t)$ can be interpreted as a unilateral probability density function in τ, $\tau \in \mathbb{R}_+$ for each value of t, $t > 0$. Then the general subordination principle is valid:

$$S_{\alpha,\beta,n}(t)\varphi = \int_{\mathbb{R}^n} G_{\alpha,\beta,n}(\zeta,t)\varphi(\mathrm{x}-\zeta)\, d\zeta \ =$$

$$\int_{\mathbb{R}^n} \int_0^\infty \Phi(\tau,t) G_{\hat\alpha,\hat\beta,n}(\zeta,\tau)\, d\tau\, \varphi(\mathrm{x}-\zeta)\, d\zeta =$$

$$\int_0^\infty \Phi(\tau,t) \int_{\mathbb{R}^n} G_{\hat\alpha,\hat\beta,n}(\zeta,\tau)\, \varphi(\mathrm{x}-\zeta)\, d\zeta\, d\tau \ = \ \int_0^\infty \Phi(\tau,t) S_{\hat\alpha,\hat\beta,n}(\tau)\varphi\, d\tau.$$

Subordination principle for the fundamental solution

$$G_{\alpha,\beta,n}(\mathrm{x},t) = \frac{1}{\alpha}\frac{t^{-\frac{\beta n}{\alpha}}}{(4\pi)^{\frac{n}{2}}}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} K_{\alpha,\beta,n}(s) z^{-s}\, ds\,, \quad z = \frac{|\mathrm{x}|}{2t^{\frac{\beta}{\alpha}}},$$

$$K_{\alpha,\beta,n}(s) = \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{n}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(1-\frac{n}{\alpha}+\frac{s}{\alpha}\right)}{\Gamma\left(1-\frac{\beta}{\alpha}n+\frac{\beta}{\alpha}s\right)\Gamma\left(\frac{n}{2}-\frac{s}{2}\right)}.$$

For the fundamental solution to the conventional diffusion equation ($\alpha = 2,\ \beta = 1$):

$$G_{2,1,n}(\mathrm{x},t) = \frac{1}{2}\frac{t^{-\frac{n}{2}}}{(4\pi)^{\frac{n}{2}}}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} K_{2,1,n}(s) z^{-s}\, ds\,, \quad z = \frac{|\mathrm{x}|}{2t^{\frac{1}{2}}},$$

$$K_{2,1,n}(s) = \Gamma\left(\frac{s}{2}\right), \quad G_{2,1,n}(\mathrm{x},t) = \frac{1}{(\sqrt{4\pi t})^n}\exp\left(-\frac{|\mathrm{x}|^2}{4t}\right).$$

Subordination principle for the fundamental solution

The general kernel function $K_{\alpha,\beta,n}(s)$ can be represented as follows:

$$K_{\alpha,\beta,n}(s) = K_{2,1,n}(s) \times \Phi^*_{\alpha,\beta,n}(s),$$

where

$$\Phi^*_{\alpha,\beta,n}(s) = \frac{\Gamma\left(\frac{n}{\alpha} - \frac{s}{\alpha}\right)\Gamma\left(1 - \frac{n}{\alpha} + \frac{s}{\alpha}\right)}{\Gamma\left(1 - \frac{\beta}{\alpha}n + \frac{\beta}{\alpha}s\right)\Gamma\left(\frac{n}{2} - \frac{s}{2}\right)}.$$

Subordination principle for the fundamental solution

Because of the Mellin convolution theorem, we get the integral representation

$$G_{\alpha,\beta,n}(\mathrm{x},t) = \frac{1}{\alpha}\frac{t^{-\frac{\beta n}{\alpha}}}{(4\pi)^{\frac{n}{2}}}\int_0^{\infty}\Phi_{\alpha,\beta,n}(\tau)\,\tilde{G}_{2,1,n}\left(\frac{z}{\tau}\right)\,\frac{d\tau}{\tau},\ \ z = \frac{|\mathrm{x}|}{2t^{\frac{\beta}{\alpha}}},$$

where $\Phi_{\alpha,\beta,n}(\tau)$ is the inverse Mellin integral transform of $\Phi^*_{\alpha,\beta,n}(s)$ and

$$\tilde{G}_{2,1,n}(\tau) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} K_{2,1,n}(s)\,\tau^{-s}ds.$$

Subordination principle for the fundamental solution

Main result:

Let $0 < \beta \leq 1$, $0 < \alpha \leq 2$, and $2\beta + \alpha < 4$. Then the subordination principle

$$G_{\alpha,\beta,n}(\mathrm{x}, t) = \int_0^\infty t^{-\frac{2\beta}{\alpha}} \Phi_{\alpha,\beta}(\tau\, t^{-\frac{2\beta}{\alpha}})\, G_{2,1,n}(\mathrm{x}, \tau)\, d\tau$$

is valid, where

$$G_{2,1,n}(\mathrm{x}, t) = \frac{1}{(\sqrt{4\pi t})^n} \exp\left(-\frac{|\mathrm{x}|^2}{4t}\right)$$

is the fundamental solution to the conventional diffusion equation and the kernel $t^{-\frac{2\beta}{\alpha}} \Phi_{\alpha,\beta}(\tau\, t^{-\frac{2\beta}{\alpha}})$ can be interpreted as a unilateral probability density function in τ evolving in time $(t > 0)$.

Particular cases

1) For the time-fractional diffusion equation ($\alpha = 2,\ 0 < \beta \le 1$) kernel function $\Phi_{\alpha,\beta}$ is reduced to the conventional Wright function and we arrive at the known formula

$$G_{2,\beta,n}(\mathrm{x}, t) = \int_0^\infty t^{-\beta} W_{1-\beta,-\beta}(-\tau\, t^{-\beta}) G_{2,1,n}(\mathrm{x}, \tau)\, d\tau,\ 0 < \beta < 1.$$

2) For the space-fractional diffusion equation ($\beta = 1,\ 0 < \alpha \le 2$), the kernel function $\Phi_{\alpha,\beta}$ is reduced to the conventional Wright function and we arrive at the subordination formula

$$G_{\alpha,1,n}(\mathrm{x}, t) = \int_0^\infty \tau^{-1} W_{0,-\frac{\alpha}{2}}(-\tau^{-\frac{\alpha}{2}}\, t) G_{2,1,n}(\mathrm{x}, \tau)\, d\tau,\ 0 < \alpha < 2.$$

Another subordination principle in the case $n = 2$

The same method can be used to derive the following subordination formula ($n = 2$, $\beta < \frac{\alpha}{2}$):

$$G_{\alpha,\beta,2}(\mathrm{x}, t) = \int_0^\infty t^{-\frac{2\beta}{\alpha}} W_{1-\frac{2\beta}{\alpha},-\frac{2\beta}{\alpha}}(-\tau t^{-\frac{2\beta}{\alpha}}) G_{\alpha,\alpha/2,2}(\mathrm{x}, \tau)\, d\tau,$$

where $G_{\alpha,\alpha/2,2}$ is the fundamental solution to the two-dimensional α-fractional diffusion equation, given by the formula

$$G_{\alpha,\alpha/2,2}(\mathrm{x}, t) = \frac{1}{4\pi t} \left(\frac{|x|}{2\sqrt{t}} \right)^{\alpha-2} E_{\frac{\alpha}{2},\frac{\alpha}{2}} \left(- \left(\frac{|x|}{2\sqrt{t}} \right)^{\alpha} \right)$$

in terms of the two-parameters Mittag-Leffler function. The fundamental solution $G_{\alpha,\alpha/2,2}(\mathrm{x}, t)$ can be interpreted as a pdf in x evolving in time ($t > 0$).

Yu. Luchko: A new fractional calculus model for the two-dimensional anomalous diffusion and its analysis. *Math. Model. Nat. Phenom.* **11** (2016), 1-17.

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Properties of the kernel function $\Phi_{\alpha,\beta}$

The Mellin-Barnes integral representation (inverse Mellin transform of $\Phi^*_{\alpha,\beta}(s)$) or a particular case of the Fox H-function:

$$\Phi_{\alpha,\beta}(\tau) = \frac{2}{\alpha}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma\left(\frac{2}{\alpha}-\frac{2}{\alpha}s\right)\Gamma\left(1-\frac{2}{\alpha}+\frac{2}{\alpha}s\right)}{\Gamma\left(1-\frac{2\beta}{\alpha}+\frac{2\beta}{\alpha}s\right)\Gamma\left(1-s\right)}\,\tau^{-s}\,ds.$$

Properties of the kernel function $\Phi_{\alpha,\beta}$

Series representation

$$\Phi_{\alpha,\beta}(\tau) = \begin{cases} \tau^{\frac{\alpha}{2}-1}\, W_{(1-\beta,-\beta),(\frac{\alpha}{2},\frac{\alpha}{2})}\left(-\tau^{\frac{\alpha}{2}}\right) \ \text{if} \ \ \beta < \frac{\alpha}{2}, \\ \\ \tau^{-1}\, W_{(1,\beta),(0,-\frac{\alpha}{2})}\left(-\tau^{-\frac{\alpha}{2}}\right) \ \text{if} \ \ \beta > \frac{\alpha}{2}, \\ \\ \frac{\tau^{\frac{\alpha}{2}-1}}{\pi} \sum_{k=0}^{\infty} \sin\left(\frac{\pi\alpha}{2}(k+1)\right) \left(-\tau^{\frac{\alpha}{2}}\right)^k \ \text{if} \ \ 0 < \tau < 1 \\ \\ -\frac{\tau^{-1}}{\pi} \sum_{k=0}^{\infty} \sin\left(\frac{\pi\alpha}{2}k\right) \left(-\tau^{-\frac{\alpha}{2}}\right)^k \ \text{if} \ \ \tau > 1 \end{cases},$$

with the four parameters Wright function

$$W_{(a,\mu),(b,\nu)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(a+\mu k)\Gamma(b+\nu k)}, \quad \mu, \nu \in \mathbb{R}, \ a, \ b, \ z \in \mathbb{C}.$$

Properties of the kernel function $\Phi_{\alpha,\beta}$

The four parameters Wright function

$$W_{(a,\mu),(b,\nu)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(a+\mu k)\Gamma(b+\nu k)}, \quad \mu, \nu \in \mathbb{R},\ a,\ b,\ z \in \mathbb{C}.$$

was introduced in

E.M. Wright, *The asymptotic expansion of the generalized hypergeometric function*, Journal London Math. Soc. **10** (1935), 287–293

for the positive values of the parameters μ and $\nu > 0$ (**the four parameters Wright function of the first kind**).

When $a = \mu = 1$ or $b = \nu = 1$, respectively, the four parameters Wright function is reduced to the Wright function.

Properties of the kernel function $\Phi_{\alpha,\beta}$

In the case when one of the parameters μ or ν is negative, the four parameters Wright function **(of the second kind)**

$$W_{(a,\mu),(b,\nu)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(a+\mu k)\Gamma(b+\nu k)}, \quad \mu, \nu \in \mathbb{R}, \; a, \; b, \; z \in \mathbb{C}.$$

was introduced and investigated in

Yu. Luchko, R. Gorenflo, *Scale-invariant solutions of a partial differential equation of fractional order*, Fract. Calc. Appl. Anal. **1** (1998), 63–78.

In particular, it was proved there that the function $W_{(a,\mu),(b,\nu)}(z)$ is an entire function provided that $0 < \mu + \nu, \; a, b \in \mathbb{C}$.
In the case $\mu + \nu = 0$, the four parameters Wright function is not en entire function anymore. The convergence radius of the series that defines this function is equal to one.

Properties of the kernel function $\Phi_{\alpha,\beta}$

The kernel function $\Phi_{\alpha,\beta}(\tau)$ can be also interpreted as the inverse Laplace transform of the Mittag-Leffler function $E_\beta(-\lambda^{\frac{\alpha}{2}})$:

$$E_\beta(-\lambda^{\frac{\alpha}{2}}) = \int_0^\infty \Phi_{\alpha,\beta}(\tau)\, e^{-\lambda\tau}\, d\tau.$$

This relation follows from the Mellin-Barnes representation of the kernel function $\Phi_{\alpha,\beta}(\tau)$ and is used to prove its non-negativity.
The Mittag-Leffler function E_β is defined as the following convergent series:

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + 1)}, \quad \beta > 0,\ z \in \mathbb{C}.$$

Properties of the kernel function $\Phi_{\alpha,\beta}$

The function

$$\phi(\lambda) = E_\beta\left(-\lambda^{\frac{\alpha}{2}}\right)$$

is completely monotone for $0 < \alpha \le 2$ and $0 < \beta \le 1$:

1) $\alpha = 2$: The Mittag-Leffler function $f(\lambda) = E_\beta(-\lambda)$ is completely monotone for $0 < \beta \le 1$.

2) $0 < \alpha < 2$. The function $g(\lambda) = \lambda^{\frac{\alpha}{2}}$ is a Bernstein function because its derivative $g'(\lambda) = \frac{\alpha}{2}\lambda^{\frac{\alpha}{2}-1}$ is completely monotone. A composition of a completely monotone function and a Bernstein function is completely monotone. Thus the function $\phi(\lambda) = f(g(\lambda))$ is completely monotone for $0 < \alpha < 2$, too.

Properties of the kernel function $\Phi_{\alpha,\beta}$

The famous Bernstein theorem and the representation

$$E_\beta(-\lambda^{\frac{\alpha}{2}}) = \int_0^\infty \Phi_{\alpha,\beta}(\tau)\, e^{-\lambda\tau}\, d\tau.$$

lead to non-negativity of the function

$$\Phi_{\alpha,\beta}(t) \ge 0,\ t > 0,\ 0 < \beta \le 1,\ 0 < \alpha \le 2.$$

Thus the kernel function of the subordination formula is also non-negative:

$$t^{-\frac{2\beta}{\alpha}}\Phi_{\alpha,\beta}(\tau t^{-\frac{2\beta}{\alpha}}) \ge 0,\ t > 0,\ \tau \ge 0,\ 0 < \beta \le 1,\ 0 < \alpha \le 2.$$

Properties of the kernel function $\Phi_{\alpha,\beta}$

To evaluate the integral of $t^{-\frac{2\beta}{\alpha}}\Phi_{\alpha,\beta}(\tau t^{-\frac{2\beta}{\alpha}})$ over $\mathbb{R}_+$ we use the technique of the Mellin integral transform:

$$\int_0^\infty t^{-\frac{2\beta}{\alpha}}\Phi_{\alpha,\beta}(\tau t^{-\frac{2\beta}{\alpha}})\,d\tau = \int_0^\infty \Phi_{\alpha,\beta}(\tau)\,d\tau =$$

$$\lim_{s\to 1}\frac{2}{\alpha}\frac{\Gamma\left(\frac{2}{\alpha}(1-s)\right)\Gamma\left(1-\frac{2}{\alpha}+\frac{2}{\alpha}s\right)}{\Gamma\left(1-\frac{2\beta}{\alpha}+\frac{2\beta}{\alpha}s\right)\Gamma(1-s)} = \frac{2}{\alpha}\lim_{s\to 1}\frac{\Gamma\left(\frac{2}{\alpha}(1-s)\right)}{\Gamma(1-s)} = 1.$$

Thus, the kernel function $t^{-\frac{2\beta}{\alpha}}\Phi_{\alpha,\beta}(\tau t^{-\frac{2\beta}{\alpha}})$ can be interpreted as a probability density function in τ evolving in time $t > 0$.

More details can be found in
Yu. Luchko, *Subordination principles for the multi-dimensional space-time-fractional diffusion-wave equation*. Theory of Probability and Mathematical Statistics 98, 1, 2018, 121-141.

Other recent results

Subordination principle for the multi-term time-fractional diffusion-wave equations:

E. Bazhlekova, I.B. Bazhlekov, *Subordination approach to multi-term time-fractional diffusion-wave equations*, Journal of Computational and Applied Mathematics 339 (2018), 179–192.

Subordination principle for the distributed order time-fractional diffusion-wave equations:

E. Bazhlekova, Subordination in a class of generalized time-fractional diffusion-wave equations, Fract. Calc. Appl. Anal. 21 (2018), 869–900.

Other recent results

E. Bazhlekova, Subordination principle for space-time fractional evolution equations and some applications, Integral Transform and Special Functions 30 (2019), 431–452.

Subordination principle for the abstract Cauchy problem for the space-time fractional evolution equation

$$D^{\beta}u(t) = -A^{\alpha}u(t), \quad 0 < \alpha, \beta \leq 1$$

subject to the initial conditions

$$u(0) = x,$$

where D^{β} is the Caputo fractional derivative of order β and $-A$ is a generator of a bounded C_0-semigroup in a Banach space X and A^{α} denotes the α-th fractional power of the operator A densely defined in a Banach space X, $x \in X$.

Thank you very much for your attention!

$$D_t^{\frac{\alpha}{2}} u(\mathrm{x}, t) = -(-\Delta)^{\frac{\alpha}{2}} u(\mathrm{x}, t)$$

$$G_{\alpha,\beta,n}(\mathrm{x}, t) = \int_0^\infty t^{-\frac{2\beta}{\alpha}} \Phi_{\alpha,\beta}(st^{-\frac{2\beta}{\alpha}})\, G_{2,1,n}(\mathrm{x}, s)\, ds$$

Questions and comments are welcome!