

Some inequalities for fractional Laplacians

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based on joint works with Roberta Musina

- 1) Comm. in PDEs, V. 39. 2014. N3
- 2) AIHP–AN. V. 33. 2016. N6
- 3) Bull. Math. Sci. V. 9. 2019. N1

$\Omega \in \mathbb{R}^n$ is a smooth domain.

First let us consider polyharmonic operators. The **Navier** BC for $(-\Delta)^k$, $k \in \mathbb{N}$, are defined as follows:

$$u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = \Delta^2 u|_{\partial\Omega} = \dots = \Delta^{k-1} u|_{\partial\Omega} = 0.$$

The corresponding operator $(-\Delta_\Omega)_N^k$ can be defined by its quadratic form

$$((-\Delta_\Omega)_N^k u, u) = \sum_j \lambda_j^k \cdot |(u, \varphi_j)|^2.$$

Here, λ_j and φ_j are eigenvalues and eigenfunctions of the Dirichlet Laplacian in Ω , respectively.

The **Dirichlet** BC for the operator $(-\Delta)^k$ are

$$u|_{\partial\Omega} = \frac{\partial u}{\partial \mathbf{n}}|_{\partial\Omega} = \frac{\partial^2 u}{\partial \mathbf{n}^2}|_{\partial\Omega} = \cdots = \frac{\partial^{k-1} u}{\partial \mathbf{n}^{k-1}}|_{\partial\Omega} = 0,$$

where \mathbf{n} is the unit exterior normal vector to $\partial\Omega$.

The quadratic form of corresponding operator $(-\Delta_\Omega)_D^k$ is the restriction of the quadratic form for $(-\Delta)^k$ in \mathbb{R}^n to functions supported in Ω :

$$((-\Delta_\Omega)_D^k u, u) = \int_{\mathbb{R}^n} |\xi|^{2k} |\mathcal{F}u(\xi)|^2 d\xi,$$

where \mathcal{F} is the Fourier transform.

Now for arbitrary $s > -1$ we define the “**Navier**” fractional Laplacian by the quadratic form

$$Q_{s,\Omega}^N[u] = ((-\Delta_\Omega)^s_N u, u) := \sum_j \lambda_j^s \cdot |(u, \varphi_j)|^2$$

and the “**Dirichlet**” fractional Laplacian by the quadratic form

$$Q_{s,\Omega}^D[u] = ((-\Delta_\Omega)^s_D u, u) := \int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi$$

with domains, respectively,

$$\text{Dom}(Q_{s,\Omega}^N) = \{u \in \mathcal{D}'(\Omega) : Q_s^N[u] < \infty\};$$

$$\text{Dom}(Q_{s,\Omega}^D) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \text{supp } u \subset \bar{\Omega}, Q_s^D[u] < \infty\}.$$

For $s = 1$, these two operators evidently coincide. We emphasize that, in contrast to $(-\Delta_\Omega)_N^s$, the operator $(-\Delta_\Omega)_D^s$ is not the s th power of the Dirichlet Laplacian for $s \neq 1$.

Recall that the Sobolev space $H^s(\mathbb{R}^n) = W_2^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, is the space of distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ with finite norm

$$\|u\|_s^2 = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\mathcal{F}u(\xi)|^2 d\xi,$$

Also we introduce the space

$$\widetilde{H}^s(\Omega) = \{u \in H^s(\mathbb{R}^n) : \text{supp } u \subset \overline{\Omega}\}.$$

Note that $\widetilde{H}^s(\Omega)$ coincides with $H_0^s(\Omega)$ only for $s - \frac{1}{2} \notin \mathbb{Z}$.

Caffarelli and Silvestre (2007) connected the fractional Laplacian of order $\sigma \in (0, 1)$ in \mathbb{R}^n with generalized Dirichlet-to-Neumann map. In particular, for any $u \in \widetilde{H}^\sigma(\Omega)$ the function $w_\sigma^D(x, y)$ minimizing the weighted Dirichlet integral

$$\mathcal{E}_\sigma^D(w) = \int_0^\infty \int_{\mathbb{R}^n} y^{1-2\sigma} |\nabla w(x, y)|^2 dx dy$$

over the set

$$\mathcal{W}_\sigma^D(u) = \left\{ w(x, y) : \mathcal{E}_\sigma^D(w) < \infty, \quad w|_{y=0} = u \right\},$$

satisfies

$$Q_{\sigma, \Omega}^D[u] = C_\sigma \cdot \mathcal{E}_\sigma^D(w_\sigma^D). \quad (1)$$

Moreover, $w_\sigma^D(x, y)$ is the solution of the BVP

$$-\operatorname{div}(y^{1-2\sigma} \nabla w) = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}_+; \quad w|_{y=0} = u,$$

and for sufficiently smooth u

$$(-\Delta_\Omega)_D^\sigma u(x) = -C_\sigma \cdot \lim_{y \rightarrow 0^+} y^{1-2\sigma} \partial_y w_\sigma^D(x, y), \quad x \in \Omega. \quad (2)$$

Stinga and Torrea (2010) developed this approach in quite general situation. In particular, it was shown that for any $u \in \widetilde{H}^\sigma(\Omega)$ the function $w_\sigma^N(x, y)$ minimizing the energy integral

$$\mathcal{E}_\sigma^N(w) = \int_0^\infty \int_\Omega y^{1-2\sigma} |\nabla w(x, y)|^2 dx dy$$

over the set

$$\mathcal{W}_{\sigma, \Omega}^N(u) = \{w(x, y) \in \mathcal{W}_\sigma^D(u) : w|_{x \in \partial\Omega} = 0\},$$

satisfies

$$Q_{\sigma, \Omega}^N[u] = C_\sigma \cdot \mathcal{E}_\sigma^N(w_\sigma^N). \quad (3)$$

Moreover, $w_\sigma^N(x, y)$ is the solution of the BVP

$$-\operatorname{div}(y^{1-2\sigma} \nabla w) = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+; \quad w|_{y=0} = u; \quad w|_{x \in \partial\Omega} = 0,$$

and for sufficiently smooth u it turns out that

$$(-\Delta_\Omega)_N^\sigma u(x) = -C_\sigma \cdot \lim_{y \rightarrow 0^+} y^{1-2\sigma} \partial_y w_\sigma^N(x, y). \quad (4)$$

In a similar way, we connect negative fractional Laplacians of order $-\sigma \in (-1, 0)$ with generalized Neumann-to-Dirichlet map. Namely, let $u \in \text{Dom}(Q_{-\sigma, \Omega}^D)$. Consider the problem of minimizing the functional

$$\tilde{\mathcal{E}}_{-\sigma}^D(w) = \int_0^\infty \int_{\mathbb{R}^n} y^{1-2\sigma} |\nabla w(x, y)|^2 dx dy - 2 \langle u, w|_{y=0} \rangle$$

over the set $\mathcal{W}_{-\sigma}^D$, that is closure of smooth functions on $\mathbb{R}^n \times \bar{\mathbb{R}}_+$ with bounded support, with respect to $\mathcal{E}_\sigma^D(\cdot)$. We recall that u can be considered as a compactly supported functional on $H^\sigma(\mathbb{R}^n)$, and thus the duality $\langle u, w|_{y=0} \rangle$ is well defined.

Denote the minimizer of $\tilde{\mathcal{E}}_{-\sigma}^D$ by $w_{-\sigma}^D(x, y)$. Then formulae (1) and (2) imply relations

$$Q_{-\sigma, \Omega}^D[u] = -C_\sigma^{-1} \cdot \tilde{\mathcal{E}}_{-\sigma}^D(w_{-\sigma}^D); \quad (-\Delta_\Omega)_D^{-\sigma} u(x) = C_\sigma^{-1} w_{-\sigma}^D(x, 0), \quad x \in \Omega, \quad (5)$$

that give the “dual” variational characterization of $(-\Delta_\Omega)_D^{-\sigma}$.

Remark. Note that for sufficiently smooth u the function $w_{-\sigma}^D$ solves the Neumann problem

$$-\operatorname{div}(y^{1-2\sigma}\nabla w) = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+; \quad \lim_{y \rightarrow 0^+} y^{1-2\sigma} \partial_y w = -u.$$

Analogously, formulae (3) and (4) imply the “dual” variational characterization of $(-\Delta_\Omega)_N^{-\sigma}$. Namely, the function $w_{-\sigma}^N(x, y)$ minimizing the functional

$$\tilde{\mathcal{E}}_{-\sigma}^N(w) = \int_0^\infty \int_\Omega y^{1-2\sigma} |\nabla w(x, y)|^2 dx dy - 2 \langle u, w|_{y=0} \rangle$$

over the set

$$\mathcal{W}_{-\sigma, \Omega}^N(u) = \{w(x, y) \in \mathcal{W}_{-\sigma}^D : w|_{x \notin \Omega} = 0\},$$

satisfies

$$Q_{-\sigma, \Omega}^N[u] = -C_\sigma^{-1} \cdot \tilde{\mathcal{E}}_{-\sigma}^N(w_{-\sigma}^N); \quad (-\Delta_\Omega)_N^{-\sigma} u(x) = C_\sigma^{-1} w_{-\sigma}^N(x, 0). \quad (6)$$

Theorem 1. Let $s > -1$, $s \notin \mathbb{N}_0$. Then for $u \in \text{Dom}(Q_{s,\Omega}^D)$, $u \not\equiv 0$, the following relations hold:

$$Q_{s,\Omega}^N[u] > Q_{s,\Omega}^D[u], \text{ if } 2k < s < 2k + 1, \quad k \in \mathbb{N}_0; \quad (7)$$

$$Q_{s,\Omega}^N[u] < Q_{s,\Omega}^D[u], \text{ if } 2k - 1 < s < 2k, \quad k \in \mathbb{N}_0. \quad (8)$$

1. Let $s = \sigma \in (0, 1)$. We construct extensions w_σ^D and w_σ^N as described above.

We evidently have $\mathcal{W}_{\sigma,\Omega}^N \subset \mathcal{W}_\sigma^D$ and $\mathcal{E}_\sigma^N = \mathcal{E}_\sigma^D|_{\mathcal{W}_{\sigma,\Omega}^N}$. Therefore, formulae (1) and (3) provide

$$Q_{s,\Omega}^N[u] = C_\sigma \cdot \inf_{w \in \mathcal{W}_{\sigma,\Omega}^N} \mathcal{E}_\sigma^N(w) \geq C_\sigma \cdot \inf_{w \in \mathcal{W}_\sigma^D} \mathcal{E}_\sigma^D(w) = Q_{s,\Omega}^D[u].$$

To complete the proof, we observe that for $u \not\equiv 0$ the function w_σ^N cannot be a solution of the homogeneous elliptic equation in the whole half-space since such a solution is analytic in the half-space. Thus, it cannot provide $\inf_{w \in \mathcal{W}_\sigma^D} \mathcal{E}_\sigma^D(w)$, and (7) follows.

2. Let $-1 < s < 0$. We define $\sigma = -s \in (0, 1)$ and construct extensions $w_{-\sigma}^D$ and $w_{-\sigma}^N$. All arguments above hold, but the inequality is reversed by the “ $-$ ” sign in (5) and (6).

3. Now let $s > 1$, $s \notin \mathbb{N}$. We put $k = \lfloor \frac{s-1}{2} \rfloor$ and define for $u \in \widetilde{H}^s(\Omega)$

$$v = (-\Delta)^k u \in \widetilde{H}^{s-2k}(\Omega), \quad s - 2k \in (-1, 0) \cup (0, 1).$$

Note that $v \neq 0$ if $u \neq 0$. Then we have

$$Q_{s,\Omega}^N[u] = Q_{s-2k,\Omega}^N[v], \quad Q_{s,\Omega}^D[u] = Q_{s-2k,\Omega}^D[u],$$

and the conclusion follows from cases 1 and 2.

Remark. R. Frank and L. Geisinger (2016; preprint 2013) proved a general result which gives Theorem 1 for $s \in (0, 1)$ with \geq sign.

We also obtain a pointwise comparison result.

Theorem 2. Let $0 < |s| < 1$, and let $u \in \text{Dom}(Q_{s,\Omega}^D)$, $u \geq 0$, $u \not\equiv 0$. Then the following relations hold:

$$(-\Delta_{\Omega})_N^s u > (-\Delta_{\Omega})_D^s u, \text{ if } 0 < s < 1;$$

$$(-\Delta_{\Omega})_N^s u < (-\Delta_{\Omega})_D^s u, \text{ if } -1 < s < 0.$$

The inequalities are understood in the sense of distributions.

Remark. M. Fall (preprint, 2012) proved this for $s = \frac{1}{2}$ and for smooth u .

We prove Theorem 2 for $s = \sigma \in (0, 1)$. First, let $u \in C_0^\infty(\Omega)$. We construct extensions w_σ^D and w_σ^N described above. Since w_σ^D vanishes at infinity, $w_\sigma^D(x, t) > 0$ for $t > 0$ by the maximum principle. Then the strong maximum principle gives

$$W := w_\sigma^D - w_\sigma^N > 0 \quad \text{in } \Omega \times \mathbb{R}_+.$$

After changing of the variable $t = y^{2\sigma}$ the function $W(x, t)$ solves

$$\Delta_x W + 4s^2 t^{\frac{2s-1}{s}} W_{tt} = 0 \quad \text{in } \Omega \times \mathbb{R}_+; \quad W|_{t=0} = 0. \quad (9)$$

The differential operator in (9) satisfies the assumptions of the boundary point lemma (L.I. Kamynin & B.N. Himchenko, 1975) at any point $(x_0, 0) \in \Omega \times \{0\}$. Thus, we have

$$\liminf_{y \rightarrow 0^+} y^{1-2\sigma} \partial_y W(x, y) = 2\sigma \cdot \liminf_{t \rightarrow 0^+} \partial_t W(x, t) > 0.$$

For $u \in \widetilde{H}^s(\Omega)$ the statement holds by approximation argument.

The case $s < 0$ is managed in a similar way.

Now we consider an evident identity:

$$(-\Delta u, u) = \int_{\Omega} |\nabla u|^2 = (-\Delta |u|, |u|).$$

What happens for fractional Laplacians?

Theorem 3. Let $0 < |s| < 1$, and let $u \in \text{Dom}(Q_{s,\Omega}^D)$ be not of constant sign in Ω . Then the following relations hold:

$$Q_{s,\Omega}^{N/D}[u] > Q_{s,\Omega}^{N/D}[|u|], \quad \text{if } 0 < s < 1;$$

$$Q_{s,\Omega}^{N/D}[u] < Q_{s,\Omega}^{N/D}[|u|], \quad \text{if } -1 < s < 0.$$

We prove Theorem 3 for $s = \sigma \in (0, 1)$ and for the Navier Laplacian. We construct extension w_{σ}^N described above. It is evident that $|w_{\sigma}^N|$ is an admissible extension of $|u|$. Therefore, formula (3) provides

$$Q_{s,\Omega}^N[u] = C_{\sigma} \cdot \mathcal{E}_{\sigma}^N(w_{\sigma}^N) = C_{\sigma} \cdot \mathcal{E}_{\sigma}^N(|w_{\sigma}^N|) \geq C_{\sigma} \cdot \inf_w \mathcal{E}_{\sigma}^N(w) = Q_{s,\Omega}^N[|u|].$$

To complete the proof, we observe that if u changes the sign then $|w_{\sigma}^N|$ cannot be a solution of the homogeneous elliptic equation in the half-cylinder due to the maximum principle. Thus, it cannot provide $\inf_w \mathcal{E}_{\sigma}^N(w)$, and the statement follows.

The Dirichle case and the case $s < 0$ are managed in the same way.

However, the operator $u \mapsto |u|$ is bounded in $\widetilde{H}^s(\Omega)$ not only for $|s| \leq 1$ but also for $1 < s < 3/2$. So, the quantities $Q_{s,\Omega}^{N/D}[|u|]$ are well defined for $u \in \widetilde{H}^s(\Omega)$, $1 < s < 3/2$.

Theorem 4. Let $1 < s < 3/2$, and let $u \in \text{Dom}(Q_{s,\Omega}^D)$ be not of constant sign in Ω . Then the following relation holds:

$$Q_{s,\Omega}^D[u] < Q_{s,\Omega}^D[|u|].$$

The proof is much more complicated than for $|s| < 1$.

Conjecture. The same inequality holds for $Q_{s,\Omega}^N$.