Hilbert-type inequalities with non-homogeneous kernel: another view

Tibor K. Pogány

Óbuda University, Institute of Applied Mathematics, 1034 Budapest, Hungary
University of Rijeka, Faculty of Maritime Studies, H–51000 Rijeka, Croatia

Seminar on Analysis, Differential Equations and Mathematical Physics, RMC, SFU, Rostov na Donu, Russia

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Dedicated to the 186th birth anniversary of Émile Léonard Mathieu
(Metz: May 15, 1835 - † Nancy: October 19, 1890)

and to the 156th birth anniversary of Eugène Cahen
(Paris: March 18, 1865 - † April 11, 1941)

Short invitation to Mathieu series

In a set of articles

- É. Mathieu, Mémoire sur des intégrations relatives a l'équilibre d'élasticité, *J. Ecole Polytéchnique* **29** (1880), 163–206.
- É. Mathieu, Mémoire sur l'équilibre d'élasticité d'un prisme rectangle, *J. Ecole Polytéchnique* **30** (1881), 173–196.
- É. Mathieu, Sur léquilibre d'élasticité d'un prisme rectangle, *C. R. Acad. Sci. Paris* **90** (1890), 1272–1274.

Mathieu investigated the banding and vibration for clamped rectangular plates and membranes of different shape and prisms by the biharmonic DE $\Delta^2 f = g$ in modeling of elasticity of plates and membranes (Lauricella, Koialovich, Meleshko).

- T. K. Pogány, H. M. Srivastava, Ž. Tomovski, Some families of Mathieu **a**-series and alternating Mathieu **a**-series, *Appl. Math. Comput.* **173** (2006), No. 1, 69–108.
- R. K. Parmar, G. V. Milovanović, T. K. Pogány. Multi-parameter Mathieu, and alternating Mathieu series. *Appl. Math. Comput.* **400** (2021), (to appear), Article ID 126099.

In the monograph

- Émile Léonard Mathieu, *Traité de physique mathématique* **VI-VII**, Paris: Gauthier Villars, 1890.

was completed the overview of previously mentioned goal for what he introduced the series

$$S(r) = \sum_{n \geq 1} \frac{2n}{(n^2 + r^2)^2}, \qquad r > 0. \tag{1}$$

Among others $S(r)$ has been investigated by Alzer, Bai-Ni Guo, Berg, Draščić, Cerone, Diananda, Elbert, Emersleben, Feng Qi, Frontzak, Gavrea, Guo, Jankov, Lampret, Makai, Mortici, Milovanović, Russell, Srivastava, Tomovski, Trenčevski, Wangs and myself among others; for instance by the generalization

$$S_\mu(r, \alpha, \beta, \mathbf{a}) = \sum_{n \geq 1} \frac{a_n^\beta}{(a_n^\alpha + r^2)^\mu} \tag{2}$$

introduced by Feng Qi [25] in 2003 $r, \mu, \alpha, \beta, \mathbf{a} = (a_n) > 0$, who posed an "Open Problem" asking about the integral representation of (2).

Main consideration directions:

(i) Integral expressions for (1). The Emersleben (1952) formula reads

$$S(r) = \sum_{n\geq 1} \frac{2n}{(n^2+r^2)^2} = \frac{1}{r}\int_0^\infty \frac{t\sin(rt)}{e^t-1}\,dt;$$

further

$$S_\mu(r,\alpha,1,\mathbf{a}) = \mu \int_{a_1^\alpha}^\infty \int_0^{[a^{-1}(t^{1/\alpha})]} \frac{a(u)+a'(u)\{u\}}{(t+r^2)^{\mu+1}}\,\mathrm{d}t\mathrm{d}u\,,$$

where $a'(x) > 0$, Pogány(2004).

(ii) Sharp bounds. E.g. the bilateral Alzer–inequality:

$$\frac{1}{2r^2+\kappa_1} < S(r) < \frac{1}{2r^2+\kappa_2},$$

where $\kappa_1^{-1} = \zeta(3)$ is the Apéry's constant and $\kappa_2 = 1/3$. Let us mention its surprising interpolation:

$$\frac{1}{r^2+\kappa_1} \leq 2\int_1^\infty \frac{[\sqrt{t}]^2}{(r^2+t)^3}\,\mathrm{d}t \leq S(r)\,.$$

Here $r \in [r_1 \approx 0.394443, r_2 \approx 5.04572]$ and

$$S(r) < 4\int_1^\infty \frac{[\sqrt{t}]}{(r^2+t)^3}\,dt + 2\int_1^\infty \frac{[\sqrt{t}]^2}{(r^2+t)^3}\,dt \le \frac{1}{r^2+\kappa_2}$$

where $r \in \mathbb{R}_+ \setminus [r_3 \approx 0.660463, r_4 \approx 2.74663]$, Draščić & Pogány (2004).

(iii) Convergence issues, multiple series, alternating series, Mathieu–type series in which (a_n) is built by special functions of hypergeometric type:

1. Gaußian ${}_2F_1$,
2. Generalized hypergeometric ${}_pF_q$,
3. Fox–Wright hypergeometric ${}_p\Psi_q$,
4. Meijer G,
5. Fox H etc.

(iv) Applications in the theories of Kapteyn, Neumann, Schlömilch, Dini series, Butzer–Flocke–Hauß complete Ω–functions, Eisenstein, Hilbert-Eisenstein series; Hurwitz–Lerch Zeta function; Hilbert inequalities etc.

Mathieu $(\mathbf{a}, \boldsymbol{\lambda})$–series

Let us define *Mathieu* $(\mathbf{a}, \boldsymbol{\lambda})$*–series*, reads as follows:

$$S_\mu(\varrho, \mathbf{a}, \boldsymbol{\lambda}) = \sum_{n\geq 0} \frac{a(n)}{(\lambda(n)+\varrho)^\mu}$$

$$\Big(\varrho, \mu > 0, \quad \mathbf{a} := (a(n))_{n\in\mathbb{N}_0}, \quad \boldsymbol{\lambda} := (\lambda(n))_{n\in\mathbb{N}_0} \uparrow \infty\Big).$$

The first important result concerning $S_\mu(\varrho, \mathbf{a}, \boldsymbol{\lambda})$ is the related integral representation (4) below, derived with the aid of the Cahen's formula, that is the associated Laplace integral expression of the associated Dirichlet series; compare for latter section V. of the monograph

- J., Karamata, *Teorija i praksa Stiltjesova integrala*, Beograd: SANU, Posebna izdanja **CLIV/I**, Matematički institut, 1949.

Cahen's formula

In his study upon the convergence of complex parameter Dirichlet series, but mainly in his massive memoir

- E. Cahen, Sur la fonction $\zeta(s)$ de Riemann et sur le fonctions analogues, *Ann. Sci. Ec. Norm. Sup. 3* (1894), **11**, 75–164.

Eugéne Cahen derived the Laplace–integral representation formula for the Dirichlet series. Namely, letting $(\lambda_n) > 0$ monotone incerasing divergent to ∞ and the DS's parameter $\Re(x) > 0$, we have

$$\sum_{n\geq 1} a_n \mathrm{e}^{-\lambda_n x} = x \int_0^\infty \mathrm{e}^{-xt} \mathscr{D}_{\mathbf{a}}(t)\, \mathrm{d}t\,. \tag{3}$$

Here

$$\mathscr{D}_{\mathbf{a}}(t) = \sum_{n:\ \lambda_n \leq t} a_n$$

stands for the so–called *counting function*.

In turn, Cahen didn't gave a strict proof of this result, which appeared in 1908 by O. Perron. Further reading: Hardy–M. Riesz (1915), Karamata (1949).

Theorem 1 (Pogány(2004))

Let $\lambda(n) > 0$ be a monotone increasing sequence with $\lim_{n\to\infty} \lambda(n) = \infty$, while $\varrho, \mu > 0, a \in C^1[0,\infty)$, $a > 0$. Then

$$S_\mu(\varrho, \mathbf{a}, \boldsymbol{\lambda}) = \frac{a(0)}{\varrho^\mu} + \mu \int_0^\infty \int_0^{[\lambda^{-1}(t)]} \frac{a(u) + a'(u)\{u\}}{(\varrho + t)^{\mu+1}}\, dtdu. \tag{4}$$

Theorem 2 (Pogány(2004))

Under the same assumptions $S_\mu(\varrho, \mathbf{a}, \boldsymbol{\lambda}) \in [A, A + B)$, where

$$A := \frac{a(0)}{\varrho^\mu} + \mu \int_0^\infty \int_0^{[\lambda^{-1}(t)]} \frac{a(u)dtdu}{(\varrho + t)^{\mu+1}}\,,$$

$$B := \mu \int_0^\infty \frac{a([\lambda^{-1}(t)])}{(\varrho + t)^{\mu+1}}\, dt.$$

Consider $S_{p+1}(r^2, \mathbf{a}, \mathbf{a}^2) \equiv S_{p+1}(r, \mathbf{a})$ in Theorem 1 which turns out to be the so–called *Mathieu* **a**–*series*:

$$S_{p+1}(r, \mathbf{a}) = \sum_{n \geq 0} \frac{a(n)}{(a^2(n) + r^2)^{p+1}} \qquad \big(r, p+1 > 0\big).$$

The related result is

Theorem 3 (Pogány(2004))

Assume $r, p > -1$ *and*

$$\mathbf{a}: \ 0 < a(1) < \cdots < a(n) < \cdots \uparrow \infty.$$

Then we have

$$S_{p+1}(r, \mathbf{a}) = \frac{\sqrt{\pi}}{2(2r)^{p-1/2}\Gamma(p+1)} \int_0^\infty \int_{a(1)}^\infty e^{-xt} x^{p+3/2} [a^{-1}(t)]\, J_{p-1/2}(rx) dx dt\,, \tag{5}$$

where

$$J_\nu(z) = \sum_{m \geq 0} \frac{(-1)^m \big(z/2\big)^{\nu+2m}}{\Gamma(\nu+m+1)\, m!}$$

is the Bessel function of the first kind of order ν.

By this kind derivation method we've been connnect J_ν and the integral representation result in Theorem 3 (and simultaneously to the hypergeometric ${}_1F_2$); the glue was the Fredholm integral equation of the first kind with non–degenerate kernel and the Mellin transform.

Theorem 4 (Draščić and Pogány(2005))

The convolution type Fredholm IE of the first kind with non–degenerate kernel, reads as follows

$$\int_0^\infty x^{\nu+1}\psi(rx)\mathfrak{L}_x[v^{2/\alpha}]\,\mathrm{d}x = g_\nu(r,\alpha) \tag{6}$$

possesses a particular solution

$$\psi(x) = J_\nu(x) + h(x), \quad h \perp x^{\nu+1}\mathfrak{L}_x\left[v^{2/\alpha}\right],$$

if and only if

$$g_\nu(r,\alpha) = \frac{(2r)^\nu\Gamma(\nu+5/2)}{\sqrt{\pi}(\alpha+2)}\int_1^\infty \frac{1}{(r^2+t)^{\nu+5/2}} \times\left(4[t^{1/\alpha}]^{\alpha/2+1} + \alpha(\alpha+2)\int_1^{[t^{1/\alpha}]} u^{\alpha/2-1}\{u\}\mathrm{d}u\right)\mathrm{d}t\,.$$

As to an example for $h(x)$ we can construct several examples, since it is not an unique function. For intance, let $\xi_{\nu,\alpha} \geq 0$ a non–negative rv on a standard probability space $(\Omega, \mathfrak{F}, \mathsf{P})$ having PDF

$$f_{\nu,\alpha}(x) = \begin{cases} \dfrac{x^{\nu+1}\mathfrak{L}_x[v^{2/\alpha}]}{\Gamma(\nu+2)\mathfrak{M}_{-\nu-1}([v^{2/\alpha}])} & x > 0, \\ 0 & x \leq 0 \end{cases}.$$

Since P is monotone, the unique median $x_{0.5}$ turns out to be the solution of the equation

$$\mathsf{P}\{\xi_{\nu,\alpha} \leq x_{0.5}\} = \int_0^{x_{0.5}} f_{\nu,\alpha}(x)dx = \frac{1}{2}$$

Then

$$h(x) = \begin{cases} 1 & x \in [0, x_{0.5}), \\ -1 & x \geq x_{0.5}. \end{cases}$$

Point out that (6) results as the conjunction of (5) and (4) for some suitable parametrization; in fact (5) is a "relative" of the Cerone–Lenard [4] formula which is a consequence of the Gegenbauer expression

$$\int_0^\infty e^{-px} x^{\nu+1} J_\nu(qx)dx = \frac{2p(2q)^\nu \Gamma(\nu+3/2)}{\sqrt{\pi}(p^2+q^2)^{\nu+3/2}} \qquad (\nu+1>0, p>0);$$

which is actually the Laplace–Mellin transform of J_ν). Solving (6) by the inverse Mellin–transform, we get

$$J_\nu(x) = \frac{2^{\nu-2}\Gamma(\nu+\frac{5}{2})}{\pi^{3/2} i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{B\left(\frac{\nu+s}{2}, \frac{\nu-s+5}{2}\right) \int_1^\infty \frac{[\sqrt{t}]([\sqrt{t}]+1)}{t^{(\nu-s+5)/2}}\, dt}{\Gamma(\nu-s+2)\zeta(\nu-s+2)x^s}\, \mathrm{d}s$$

compare to [5, Eq. (15)]. This formula is NOT contained at the

$\texttt{http}: //\texttt{functions.wolfram.com}/\texttt{BesselAiryStruveFunctions}/\texttt{BesselJ}$

page where 396 formulae are exposed.

Theorem 5 (Pogány *et al.* (2006))

Set $\nu + 1 > 0$. *Then for all* $c_0 \in (-\nu, \nu + 1)$ *we have*

$$J_\nu(x) = \frac{2^{\nu-1}}{\pi^{3/2} i} \int_{c_0 - i\infty}^{c_0 + i\infty} x^{-\sigma} \frac{\Gamma\big(\frac{\nu+\sigma}{2}\big)\,\Gamma\big(\frac{\nu-\sigma+5}{2}\big)\, \int_1^\infty t^{-(\nu-\sigma+5)/2} \widetilde{H}_J(t)\, dt}{\Gamma(\nu-\sigma+2)\,\eta(\nu-\sigma+2)}\, \mathrm{d}\sigma,$$

where

$$\widetilde{H}_J(t) = \frac{[\sqrt{t}]([\sqrt{t}]+1)}{2} - 2\left[\frac{[\sqrt{t}]}{2}\right]\left(\left[\frac{[\sqrt{t}]}{2}\right] + 1\right).$$

Next, introduce the differential operator

$$\mathfrak{d}_x^{\mathfrak{q}} := \mathfrak{q} + \{\mathfrak{q}x\}\frac{\partial}{\partial x}\,.$$

Theorem 6 (Pogány *et al.* (2006))

Assume $\mu, \gamma, 2\mu - \gamma > 0$. *For all* $c_\mu \in (0, 2\mu - 1) \cap \mathbf{R}_\sigma^F$ *we have*

$$
{}_1F_2\Big(\mu; \mu - \frac{\gamma}{2}, \mu - \frac{\gamma - 1}{2}; -\frac{x^2}{4}\Big) = \frac{\mu\,\Gamma(2\mu - \gamma)}{2\pi i} \int_{c_0 - i\infty}^{c_0 + i\infty} \frac{B\big(\frac{\sigma}{2}, \mu - \frac{\sigma}{2} + 1\big) \int_1^\infty t^{-(\mu - \sigma/2 + 1)}\, \widetilde{H}_F(t)\, dt}{\Gamma(2\mu - \sigma - \gamma)\, \eta(2\mu - \sigma - \gamma)\, x^\sigma}\, \mathrm{d}\sigma\,,
$$

where

$$
\widetilde{H}_F(t) = \int_0^{[t^{1/(2\gamma)}]} \mathfrak{d}_u^1 u^\gamma du - 2 \int_0^{2[[t^{1/(2\gamma)}]/2]} \mathfrak{d}_u^{1/2} u^\gamma \mathrm{d}u,
$$

while

$$
\mathbf{R}_\sigma^F = \{\sigma \in \mathbb{C} : (2\mu - \gamma + \ell_1 - \Re\{\sigma\})(2\mu + 2 + \ell_2 - \Re\{\sigma\}) \neq 0,\ (\ell_1, \ell_2) \in \mathbb{N}_0^2\}\,.
$$

Riemann ζ and Dirichlet η

Consider now the integral expressions derived for Mathieu series as functions in $r \to 0$. In some special cases we arrive at Riemann Zeta and Dirichlet Eta. Actually

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}\,; \quad \eta(\sigma) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n^\sigma} \qquad (\Re\{s\} > 1, \Re\{\sigma\} > 0)$$

and $\eta(s) = (1-2^{1-s})\zeta(s)$.

Theorem 7 (Pogány *et al.* (2006))

Let $r, \mu > 0$ *and let* $a\colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ *be positiv monotone decreasing. Then we have*

$$\int_{a_1}^\infty \frac{[a^{-1}(x)]}{x^\kappa}\,dx = \frac{1}{\kappa-1}\sum_{n=1}^\infty a_n^{-\kappa+1},$$

$$\int_{a_1}^\infty \frac{\sin^2\left(\frac{\pi}{2}[a^{-1}(x)]\right)}{x^\kappa}\,dx = \frac{1}{\kappa-1}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{a_n^{\kappa-1}}\,;$$

κ *mutually ensure the convergence of involved series.*

Corollary 1 (Pogány *et al.* (2006))

For $s-1, \sigma > 0$, *it holds*

$$\zeta(s) = s \int_1^\infty \frac{[x]}{x^{s+1}}\, dx,$$

$$\eta(\sigma) = \sigma \int_1^\infty \frac{\sin^2\left(\frac{\pi}{2}[x]\right)}{x^{\sigma+1}}\, dx$$

$$(1-2^{1-s}) \int_1^\infty \frac{[x]}{x^{s+1}}\, dx = \int_1^\infty \frac{\sin^2\left(\frac{\pi}{2}[x]\right)}{x^{s+1}}\, dx\,.$$

Interesting formula can be derived for the celebrated "Apéry constant", also for $\eta(3)$:

Corollary 2 (Pogány *et al.* (2006))

$$\zeta(3) = \int_1^\infty \frac{[\sqrt{t}]([\sqrt{t}]+1)}{t^3}\, dt,$$

$$\frac{1}{2}\Big(\zeta(3) - \frac{1}{2}\eta(3)\Big) = \int_1^\infty \left[\frac{[\sqrt{t}]}{2}\right] \left(\left[\frac{[\sqrt{t}]}{2}\right] + 1 \right) \frac{dt}{t^3}\,.$$

Non-homogeneous kernel discrete Hilbert type inequalities

Let ℓ_p the space of complex sequences $\mathbf{x} = (x_n)_{n\geq 1}$ with the usual finite $\|\mathbf{x}\|_p := \big(\sum_{n\geq 1}|x_n|^p\big)^{1/p}$ norm endowed. The Hilbert "double series theorem", say, means

$$\sum_{m,n\geq 1}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\|\mathbf{a}\|_p\,\|\mathbf{b}\|_q \qquad \big(p>1,\ p^{-1}+q^{-1}=1\big), \tag{7}$$

where positive $\mathbf{a} = \big(a_n\big)_{n\geq 1} \in \ell_p, \mathbf{b} = \big(b_n\big)_{n\geq 1} \in \ell_q$ and

$$\mathrm{B}(p, 1-p) = \frac{\pi}{\sin(\pi/p)}$$

is the best constant, see Hardy–Litlewood–Pólya [10, p. 253].

When $p > 1$, $p^{-1} + q^{-1} = 1$, $2 - \min\{p, q\} < \lambda \leq 2$, then by Bi Cheng Yang [31, Eq.(5)], there holds true

$$\sum_{m,n\geq 1} \frac{a_m b_n}{(m+n)^\lambda} < k_\lambda(p) \, \|\mathbf{n}^{(1-\lambda)/p}\mathbf{a}\|_p \, \|\mathbf{n}^{(1-\lambda)/q}\mathbf{b}\|_q, \tag{8}$$

whenever $\mathbf{n}^{(1-\lambda)/p}\mathbf{a} := \big(n^{(1-\lambda)/p} a_n\big)_{n\geq 1} \in \ell_p$, $\mathbf{n}^{(1-\lambda)/q}\mathbf{b} \in \ell_q$. Here

$$k_\lambda(p) := B\big(1 + (\lambda-2)/p, 1 + (\lambda-2)/q\big)$$

is sharp.
In short we examine the infinite bilinear form

$$\mathfrak{H}_K^{\mathbf{a},\mathbf{b}} := \sum_{m,n\geq 1} K(m,n)\, a_m b_n\,,$$

in which $\mathbf{a}, \mathbf{b} \geq 0$ while the expression $K(\cdot,\cdot)$ stands for the kernel. In the case $K(m,n) = (m^\kappa + n^\kappa)^{-\lambda}$, the kernel is homogeneous, having order $-\lambda \cdot \kappa$; $\kappa = \lambda = 1$ gives the original Hilbert's (7); $\kappa = 1$ results in Yang result (8). Non–homogeneous K was considered by Mulholland, Yang, He *at al.* and Krnić and Pečarić for instance.

Assume $\lambda, \rho \colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ monotone increasing, with

$$\lim_{x\to\infty} \left\{ \begin{array}{c} \lambda \\ \rho \end{array} \right\}(x) = \infty\,. \tag{9}$$

Restrictions $\lambda\big|_{\mathbb{N}} = \boldsymbol{\lambda} = (\lambda_n)_{n\geq 1}, \rho\big|_{\mathbb{N}} = \boldsymbol{\rho} = (\rho_n)_{n\geq 1}$ will be used to build the bilinear form

$$\mathfrak{H}^{\mathbf{a},\mathbf{b}}_{\boldsymbol{\lambda},\boldsymbol{\rho}}(\mu) := \sum_{m,n\geq 1} \frac{a_m b_n}{(\lambda_m + \rho_n)^\mu} \qquad (\mu > 0).$$

Additionally, we will apply the so–called *Hölder inequality with non–conjugate parameters*, that is when $p, q > 1$ and $p^{-1} + q^{-1} \geq 1$, introducing the new incremental parameter

$$\Delta := \frac{1}{p} + \frac{1}{q} - 1 \geq 0\,.$$

The non–conjugated parameters assumption was considered in studying Hilbert type inequalities earlier too, consult the excellent classical works by Bonsall [1] and Levin [15], for instance.

The strategy for a qualitative jump from the Mathieu $(\mathbf{a}, \boldsymbol{\lambda})$–series to Hilbert's double series/bilinear form $\mathfrak{H}_{\boldsymbol{\lambda},\boldsymbol{\rho}}^{\mathbf{a},\mathbf{b}}(\mu)$ is the following:

- Consider the series $S_\mu(\rho, \mathbf{a}, \boldsymbol{\lambda})$;
- multiply the series with a positive b_m which is the term of the sequence $\mathbf{b} = (b_m) > 0$;
- assume that $\rho = \rho_m$ is a restriction $\rho(x)\big|_{\mathbb{N}_0}$, where $\rho(x) \uparrow \infty$ monotonously;
- sum up the newly constructed single sum, now withe respect to $m \in \mathbb{N}_0$.

So the Hilbert double sum $\mathfrak{H}_{\boldsymbol{\lambda},\boldsymbol{\rho}}^{\mathbf{a},\mathbf{b}}(\mu)$. Since the background is Mathieu $(\mathbf{a}, \boldsymbol{\lambda})$–series, all results established for these series can be applied to the Hilbert double sum.

Of course, there are bounding inequalities for $S_\mu(\rho, \mathbf{a}, \boldsymbol{\lambda})$, which combined with the Hölder inequality and another tools and special functions' results will lead to the desired (sharp) bounds. Actually, the main problem with $\mathfrak{H}_{\boldsymbol{\lambda},\boldsymbol{\rho}}^{\mathbf{a},\mathbf{b}}(\mu)$ is to separate the sums, which is 'tricky' since the shape of the kernel function

$$K(m, n) = (\lambda_m + \rho_n)^{-\mu}\,.$$

However, in the case of success, we should consider the 'double(multiple)-series-problem' as the 'single-series-problem', which is already solved in different manner.

Theorem 8 (Pogány (2008))

Let $p > 1$, $p^{-1} + q^{-1} = 1$; $r \in (3 - \min\{p, q\}, 3]$, $r^{-1} + s^{-1} = 1$ *and suppose* $\mathbf{a}, \mathbf{b} \geq 0$, *and* $\left(n^{(2-r)/p} a_n^r\right)_{n \geq 1} \in \ell_p$; $\left(n^{(2-r)/q} b_n^r\right)_{n \geq 1} \in \ell_q$. *When* λ, ρ *satisfy* (9) *and* $\int_{\lambda_1}^{\infty} \frac{\left(\lambda^{-1}(x)\right)^2}{x^{s\mu/2+1}} \mathrm{d}x < \infty$; $\int_{\rho_1}^{\infty} \frac{\left(\rho^{-1}(x)\right)^2}{x^{s\mu/2+1}} \mathrm{d}x < \infty$ *then we have*

$$\mathfrak{H}_{\boldsymbol{\lambda},\boldsymbol{\rho}}^{\mathbf{a},\mathbf{b}}(\mu) = \sum_{m,n \geq 1} \frac{a_m b_n}{(\lambda_m + \rho_n)^\mu} \leq \mathsf{C}_{\boldsymbol{\lambda},\boldsymbol{\rho}} \left\| \mathbf{n}^{(2-r)/p} \mathbf{a}^r \right\|_p^{1/r} \left\| \mathbf{n}^{(2-r)/q} \mathbf{b}^r \right\|_q^{1/r}, \tag{10}$$

where the constant

$$\mathsf{C}_{\boldsymbol{\lambda},\boldsymbol{\rho}} = \mathsf{C}_{\boldsymbol{\lambda},\boldsymbol{\rho}}(p, q, r, s, \mu) := \left(\frac{\mu s}{2}(\mu s + 1)\right)^{1/s} B^{1/r}\big(1 + (r-3)/p, 1 + (r-3)/q\big)$$
$$\times \left(\int_{\lambda_1}^{\infty} \int_{\rho_1}^{\infty} \frac{[\lambda^{-1}(x)][\rho^{-1}(y)]\left([\lambda^{-1}(x)] + [\rho^{-1}(y)] + 2\right)}{(x+y)^{s\mu+2}} \mathrm{d}x\,\mathrm{d}y \right)^{1/s}$$

is sharp. Moreover in (10) *equality occurs when*

$$a_m b_n = \mathcal{K} \cdot \frac{m+n}{(\lambda_m + \rho_n)^{(s-1)\mu}} \qquad \left(\mathcal{K} \in \mathbb{R}_+;\ m, n \geq 1\right).$$

Proof.

Let $p > 1$, $p^{-1} + q^{-1} = 1$ and choose such $r > 1$, that $r \in (3 - \min\{p, q\}, 3]$, additionally s signifies the conjugated Hölder pair of r. Then we consider

$$\mathfrak{H}_{\boldsymbol{\lambda},\boldsymbol{\rho}}^{\mathbf{a},\mathbf{b}}(\mu) = \sum_{m,n\geq 1} \frac{a_m b_n}{(m+n)^{1/s}} \cdot \frac{(m+n)^{1/s}}{(\lambda_m + \rho_n)^{\mu}} \,.$$

By the Hölder inequality we conclude

$$\mathfrak{H}_{\boldsymbol{\lambda},\boldsymbol{\rho}}^{\mathbf{a},\mathbf{b}}(\mu) \leq \left(\sum_{m,n\in\mathbb{N}} \frac{a_m^r b_n^r}{(m+n)^{r-1}} \right)^{1/r} \left(\sum_{m,n\in\mathbb{N}} \frac{m+n}{(\lambda_m + \rho_n)^{s\mu}} \right)^{1/s} .$$

The first sum will be estimated by Yang's result (8), specifying $\lambda = r - 1 \in (2 - \min\{p, q\}, 2]$:

$$\sum_{m,n\geq 1} \frac{a_m^r b_n^r}{(m+n)^{r-1}} < k_{r-1}(p) \left(\sum_{n\geq 1} n^{2-r} a_n^{rp} \right)^{1/p} \left(\sum_{n\geq 1} n^{2-r} b_n^{rq} \right)^{1/q} ;$$

the second one is actually the sum of two Mathieu series, being both $(\mathbb{N}, \boldsymbol{\lambda})$–series, see (T.K. Pogány, Hilbert's double series theorem extended to the case of non-homogeneous kernels, *J. Math. Anal. Appl.* **342** (2008), No. 2, 1485–1489.)

Contributions to the non–weighted norms case

In the article

- B. Draščić Ban, T. K. Pogány, Discrete Hilbert type inequality with non-homogeneous kernel, *Appl. Anal. Discr. Math* **3** (2009), No. 1, 88-96.

our main goal was to establish simplest possible sharp upper bounds over $\mathfrak{H}_K^{\mathbf{a},\mathbf{b}}$ in terms of $\|\mathbf{a}\|_p, \|\mathbf{b}\|_q$, when the Hilbert kernel

$$K(m,n) = (\lambda_m + \rho_n)^{-\mu}$$

is non–homogeneous, that is, we are looking for a sharp estimate of the form

$$\sum_{m,n\geq 1} \frac{a_m b_n}{(\lambda_m + \rho_n)^\mu} \leq \mathrm{C}^\star \, \|\mathbf{a}\|_p \|\mathbf{b}\|_q \, .$$

This should be studied in both conjugated and non–conjugated Hölder inequality cases.
Let me draw your attention to the fact that non-homogeneous kernel $(\lambda_m + \rho_n)^{-\mu}$ generates results simultaneously for the homogeneous kernel situation too.

We are ready to state our principal inequality result.

Theorem 9 (Draščić Ban–Pogány (2009))

Suppose $p, q > 1$, $\mu > 0$, $\mathbf{a} = (a_n)_{n\in\mathbb{N}} \in \ell_p$, $\mathbf{b} = (b_n)_{n\in\mathbb{N}} \in \ell_q$ *are nonnegative sequences and* λ, ρ *are positive monotone increasing functions satisfying* (9). *Then*

$$\mathfrak{H}_{\boldsymbol{\lambda},\boldsymbol{\rho}}^{\mathbf{a},\mathbf{b}}(\mu) = \sum_{m,n\geq 1} \frac{a_m b_n}{(\lambda_m + \rho_n)^\mu} \leq \mathrm{C}_{p,q}^{\mu,\Delta}(\lambda, \rho)\, \|\mathbf{a}\|_p \|\mathbf{b}\|_q\,, \tag{11}$$

where the constant

$$\mathrm{C}_{p,q}^{\mu,\Delta}(\lambda, \rho) = \frac{q^{1/q} p^{1/p}}{\Gamma(\mu)} \int_0^\infty x^{\mu+\Delta} \left(\int_{\lambda_1}^\infty e^{-qxt} [\lambda^{-1}(t)]\, \mathrm{d}t \right)^{1/q} \times \left(\int_{\rho_1}^\infty e^{-pxu} [\rho^{-1}(u)]\, \mathrm{d}u \right)^{1/p} \mathrm{d}x\,.$$

The equality in (21) *appears for* $\lambda = \rho = \mathcal{I}$, $p = q = 2$ *when*

$$a_m\, b_n^{-1} = C\, \delta_{mn} \qquad \big(m, n \in \mathbb{N};\, C > 0\big). \tag{12}$$

Proof.

First, we transfrom the double series by means of the Gamma function. After splitting the kernel function into two Dirichlet series, we evaluate these Dirichlet series by the Hölder inequality with non– conjugated parameters $p,\, q,\, \min\{p,q\} > 1,\, p^{-1} + q^{-1} \geq 1$ [18, p. 57]. These result in

$$\begin{aligned} \mathfrak{H}^{\mathbf{a},\mathbf{b}}_{\boldsymbol{\lambda},\boldsymbol{\rho}}(\mu) &= \frac{1}{\Gamma(\mu)} \int_0^\infty x^{\mu-1} \left(\sum_{m=1}^\infty a_m e^{-\lambda_m x} \right) \left(\sum_{n=1}^\infty b_n e^{-\rho_n x} \right) \mathrm{d}x \\ &\leq \frac{\|\mathbf{a}\|_p \|\mathbf{b}\|_q}{\Gamma(\mu)} \int_0^\infty x^{\mu-1} \left(\sum_{m=1}^\infty e^{-\lambda_m q x} \right)^{1/q} \left(\sum_{n=1}^\infty e^{-\rho_n p x} \right)^{1/p} \mathrm{d}x\,. \qquad (13) \end{aligned}$$

Now, the inner–most Dirichlet series

$$\mathcal{D}_\lambda(x) = \sum_{m=1}^\infty e^{-\lambda_m q x}, \qquad \mathcal{D}_\rho(x) = \sum_{n=1}^\infty e^{-\rho_n p x}\,,$$

via (3) clearly become

$$\mathcal{D}_\lambda(x) = xq \int_0^\infty e^{-xqt} \left(\sum_{j=1}^{[\lambda^{-1}(t)]} 1 \right) \mathrm{d}t = xq \int_0^\infty e^{-xqt} [\lambda^{-1}(t)]\, \mathrm{d}t.$$

So does

$$\mathcal{D}_\rho(x) = xp \int_0^\infty e^{-xpu}[\rho^{-1}(u)]\,\mathrm{d}u\,.$$

Collecting all these expressions the upper bound in (13) becomes

$$\begin{aligned}\sum_{m,n\geq 1} \frac{a_m b_n}{(\lambda_m+\rho_n)^\mu} &\leq \frac{\|\mathbf{a}\|_p\|\mathbf{b}\|_q}{\Gamma(\mu)} \int_0^\infty x^{\mu-1}\left(qx\int_{\lambda_1}^\infty e^{-qxt}[\lambda^{-1}(t)]\,\mathrm{d}t\right)^{1/q} \\ &\qquad \times \left(px\int_{\rho_1}^\infty e^{-pxu}[\rho^{-1}(u)]\,\mathrm{d}u\right)^{1/p}\mathrm{d}x \\ &= \frac{q^{1/q}p^{1/p}\,\|\mathbf{a}\|_p\|\mathbf{b}\|_q}{\Gamma(\mu)} \int_0^\infty x^{\mu+\Delta}\left(\int_{\lambda_1}^\infty e^{-qxt}[\lambda^{-1}(t)]\mathrm{d}t\right)^{1/q} \\ &\qquad \times \left(\int_{\rho_1}^\infty e^{-pxu}[\rho^{-1}(u)]\mathrm{d}u\right)^{1/p}\mathrm{d}x\,,\end{aligned}$$

and the inequality assertion of the Theorem 9 is proved.

Finally, it remains the equality analysis in (21). Let us denote L the left–side Hilbert–type bilinear form in (21). Then making use of $a_m\, b_n^{-1} = C\,\delta_{mn}$, we get

$$L = \sum_{m,n\geq 1} \frac{a_m/b_n \ \cdot b_n^2}{(m+n)^\mu} = \sum_{m,n=1}^{\infty} \frac{C\,\delta_{mn} b_n^2}{(m+n)^\mu} = \sum_{m,n\geq 1} \frac{b_n^2}{(2m)^\mu} = \frac{\zeta(\mu)}{2^\mu}\,\|\mathbf{b}\|_2^2\,,$$

such that coincides with the right–side in (21), when (12) is fulfilled. □

Corollaries of Theorem 9

A. Taking $\lambda(x) = Ax^q$, $\rho(x) = Bx^p$, the constant becomes

$$C^{\mu,\Delta}_{p,q}(Ax^q, Bx^p) = \frac{(Aq)^{1/q}(Bp)^{1/p}}{\Gamma(\mu)} \int_0^\infty x^{\mu+\Delta}\Big(\int_A^\infty e^{-qAtx}[t^{1/q}]\mathrm{d}t\Big)^{1/q} \times \Big(\int_B^\infty e^{-pBux}[u^{1/p}]\mathrm{d}u\Big)^{1/p}\mathrm{d}x.$$

The kernel K is obviously non–homogeneous for all $p \neq q$. So we have the following

Corollary 3

Suppose $p, q > 1$, $\mu > 0$, *and* $\mathbf{a} = (a_n)_{n\in\mathbb{N}} \in \ell_p$, $\mathbf{b} = (b_n)_{n\in\mathbb{N}} \in \ell_q$ *nonnegative sequences. Then*

$$\sum_{m,n\geq 1} \frac{a_m b_n}{(Am^q + Bn^p)^\mu} \leq C^{\mu,\Delta}_{p,q}(Ax^q, Bx^p)\|\mathbf{a}\|_p\|\mathbf{b}\|_q\,.$$

B. If $\lambda = \rho = \mathcal{I}$ the kernel is homogeneous. The inequality one transforms into

$$\mathfrak{H}^{\mathbf{a},\mathbf{b}}_{\mathcal{I},\mathcal{I}}(\mu) \le \frac{\|\mathbf{a}\|_p\|\mathbf{b}\|_q}{\Gamma(\mu)} \int_0^\infty x^{\mu-1} \left(\sum_{m\ge1} e^{-mqx} \right)^{1/q} \left(\sum_{n\ge1} \mathrm{e}^{-npx} \right)^{1/p} \mathrm{d}x$$
$$= \frac{\|\mathbf{a}\|_p\|\mathbf{b}\|_q}{\Gamma(\mu)} \int_0^\infty \frac{x^{\mu-1}}{(e^{qx}-1)^{1/q}(e^{px-1})^{1/p}}\,\mathrm{d}x\,.$$

Corollary 4

Suppose $p, q > 1$, $\mu > \Delta + 1$ and let $\mathbf{a} = (a_n)_{n\in\mathbb{N}} \in \ell_p$, $\mathbf{b} = (b_n)_{n\in\mathbb{N}} \in \ell_q$ be nonnegative sequences. Then

$$\mathfrak{H}^{\mathbf{a},\mathbf{b}}_{\mathcal{I},\mathcal{I}}(\mu) \le \mathsf{C}^{\mu,\Delta}_{p,q}(\mathcal{I},\mathcal{I})\|\mathbf{a}\|_p\,\|\mathbf{b}\|_q \tag{14}$$

where

$$\mathsf{C}^{\mu,\Delta}_{p,q}(\mathcal{I},\mathcal{I}) = \frac{1}{\Gamma(\mu)} \int_0^\infty \frac{x^{\mu-1}}{(e^{qx}-1)^{1/q}(e^{px-1})^{1/p}}\,\mathrm{d}x\,.$$

C. When $\lambda(x) = \rho(x) = x^2$, the kernel $K(m^2, n^2)$ is homogeneous.

Corollary 5

Suppose $p, q > 1$, $\mu > \Delta + 1$ *and let* $\mathbf{a} = (a_n)_{n\in\mathbb{N}} \in \ell_p$, $\mathbf{b} = (b_n)_{n\in\mathbb{N}} \in \ell_q$ *be nonnegative sequences. Then*

$$\sum_{m,n\geq 1} \frac{a_m b_n}{(m^2+n^2)^\mu} \leq C_{p,q}^{\mu,\Delta}(x^2, x^2)\|\mathbf{a}\|_p\|\mathbf{b}\|_q\,.$$

In this case

$$C_{p,q}^{\mu,\Delta}(x^2, x^2) = \frac{1}{2^{\Delta+1}\Gamma(\mu)} \int_0^\infty x^{\mu-1}\big(\vartheta_3(0, e^{-px}) - 1\big)^{1/p}\big(\vartheta_3(0, e^{-qx}) - 1\big)^{1/q}\mathrm{d}x\,.$$

Here $\vartheta_3(u, \mathfrak{q}) = 1 + 2\sum_{n\geq 1} \mathfrak{q}^{n^2}\cos(2n\pi u), |q| < 1$ *stands for the Jacobi Theta.*

We point out that the case $p = q = 2$ (such that means *a fortiori* $\Delta = 0$) results in reduced constant

$$C_{2,2}^{\mu,0}(x^2, x^2) = \frac{1}{2\Gamma(\mu)} \int_0^\infty x^{\mu-1}\big(\vartheta_3(0, e^{-2x}) - 1\big)\mathrm{d}x = \frac{\zeta(2\mu)}{2^\mu}\,.$$

D. If we take $\lambda \equiv \rho$, $p = q = 2$, it depends on the nature of $\lambda(x)$ is the kernel homogeneous or not. The constant reduces to

$$\mathrm{C}_{2,2}^{\mu,0}(\lambda,\lambda) = \frac{\Gamma(\mu+1)}{\Gamma(\mu)2^\mu}\int_{\lambda_1}^\infty \frac{[\lambda^{-1}(t)]}{t^{\mu+1}}\,\mathrm{d}t = \frac{\mu}{2^\mu}\int_{\lambda_1}^\infty \frac{[\lambda^{-1}(t)]}{t^{\mu+1}}\,\mathrm{d}t\,.$$

This proves the

Corollary 6

Suppose $\mu > 0$, $\mathbf{a} = (a_n)_{n\in\mathbb{N}}, \mathbf{b} = (b_n)_{n\in\mathbb{N}} \in \ell_2$ are nonnegative sequences, λ positive monotone function such that satisfies (9), *i.e.* $\lim_{x\to\infty}\lambda(x) = \infty$. *Then*

$$\sum_{m,n\geq 1} \frac{a_m b_n}{(\lambda_m+\lambda_n)^\mu} \leq \mathrm{C}_{2,2}^{\mu,0}(\lambda,\lambda)\|\mathbf{a}\|_2\|\mathbf{b}\|_2\,.$$

E. Finally, if we specify $\lambda = \rho = \mathcal{I}$, $p = q = 2$, by Corollary 4. we get

Corollary 7

Suppose $\mu > 1$, $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$, $\mathbf{b} = (b_n)_{n\in\mathbb{N}} \in \ell_2$ are nonnegative sequences. Then

$$\sum_{m,n\geq 1} \frac{a_m b_n}{(m+n)^\mu} \leq 2^{-\mu}\,\zeta(\mu)\,\|\mathbf{a}\|_2\|\mathbf{b}\|_2\,. \tag{15}$$

Whose result is sharper?

Being $\mu > 1$, $p = q = 2$, by means of (7) the following straightforward estimates hold

$$\sum_{m,n\geq 1} \frac{a_m b_n}{(m+n)^\mu} < \sum_{m,n\geq 1} \frac{a_m b_n}{m+n} \leq \pi \|\mathbf{a}\|_2 \|\mathbf{b}\|_2 \,. \tag{16}$$

On the other hand

$$C_{2,2}^{1.1,0}(\mathcal{I},\mathcal{I}) = \frac{\zeta(1.1)}{2^{1.1}} \approx 4.937797 > \pi \,,$$

that is, (16) is sharper than (15)!
Obviously, $C_{2,2}^{\mu,0}(\mathcal{I},\mathcal{I})$ decreases with growing μ. Denote μ_0 the unique root of

$$\zeta(\mu) = \pi \, 2^\mu \,.$$

Then, for all $\mu > \mu_0$ the upper bound (15) is sharper then (16), while for $\mu \in [1, \mu_0)$, the reverse result holds true. Let us mention that $\mu_0 \approx 1.156$.

However, the used estimation method is not efficient for general non–conjugated $p, q > 1$. Indeed, we deduce

$$\sum_{m,n\geq 1} \frac{a_m b_n}{(m+n)^\mu} < \sum_{m,n\geq 1} \frac{a_m b_n}{m+n} \leq \frac{\pi}{\sin(\pi/p)} \, \|\mathbf{a}\|_p \|\mathbf{b}\|_q \,.$$

Now, by the Bernoulli inequality we get

$$\begin{aligned} C_{p,q}^{\mu,\Delta}(\mathcal{I},\mathcal{I}) &= \frac{1}{\Gamma(\mu)} \int_0^\infty x^{\mu-1} e^{-2x} \left(1 - e^{-px}\right)^{-1/p} \left(1 - e^{-qx}\right)^{-1/q} \mathrm{d}x \\ &\leq \frac{1}{\Gamma(\mu)} \int_0^\infty x^{\mu-1} e^{-2x} \left(1 - \frac{1}{p}\, e^{-px}\right)\left(1 - \frac{1}{q}\, e^{-qx}\right) \mathrm{d}x \\ &= \frac{1}{2^\mu} - \frac{1}{p(2+p)^\mu} - \frac{1}{q(2+q)^\mu} + \frac{1}{pq(2+p+q)^\mu} \\ &< \frac{1}{2^\mu}\left(1 + \frac{1}{2^\mu}\right) < \frac{\pi}{\sin(\pi/p)} \,. \end{aligned}$$

Accordingly, (14) is superior to (7) for all $\mu \geq 1$.

Final remarks

1. Except (7), no Hilbert or Hilbert type inequalities are established until now in terms of $\|\mathbf{a}\|_p, \|\mathbf{b}\|_q$. Applying different parameter Hölder inequalities in evaluating the Dirichlet series in (13), that is, first $p, q,\ p^{-1}+q^{-1} \geq 1$, then $r, s,\ r^{-1}+s^{-1} \geq 1$ we can easily (but artificially) generalize Theorem 9.

2. One of our main tasks was to express the sharp upper bound upon $\mathfrak{H}_K^{\mathbf{a},\mathbf{b}}$, K non–homogeneous, in terms of $\|\mathbf{a}\|, \|\mathbf{b}\|$. Transforming the both Dirichlet series appearing in (13) in the manner:

$$\sum_{n\geq 1} a_n \mathrm{e}^{-\lambda_n x} = \sum_{n\geq 1} \frac{a_n}{\phi(n)} \cdot \phi(n) \mathrm{e}^{-\lambda_n x}$$

ϕ certain convenient function, we can proceed to apply the Hölder inequality to the right–hand sum. Consequent new results follow easily.

3. Taking in (21) $a_n \mapsto \phi(a_n)$, $b_m \mapsto \psi(b_m)$, ϕ, ψ suitable functions, one generalizes Theorem 9 in another way.

New class of allied Hilbert double series inequalities

Now, one extends the results of Draščić Ban–Pogány (2009), [6] to a new class of inequalities by introducing and studying an auxiliary function. So, we consider the Dirichlet–series

$$\mathfrak{H}_{\boldsymbol{\lambda},\boldsymbol{\rho}}^{\mathbf{a},\mathbf{b}}(\mu;x) := \sum_{m,n\geq 1} \frac{a_m b_n}{(\lambda_m+\rho_n)^\mu}\, x^{\lambda_m+\rho_n} \qquad (x>0)$$

associated with the Hilbert's bilinear double series $\mathfrak{H}_{\boldsymbol{\lambda},\boldsymbol{\rho}}^{\mathbf{a},\mathbf{b}}(\mu;1) \equiv \mathfrak{H}_{\boldsymbol{\lambda},\boldsymbol{\rho}}^{\mathbf{a},\mathbf{b}}(\mu)$. Let the situation the same as above (regarding involved parameters) and $x>0$. Then

$$\begin{aligned}\mathfrak{H}_{\boldsymbol{\lambda},\boldsymbol{\rho}}^{\mathbf{a},\mathbf{b}}(\mu;x) &= \frac{1}{\Gamma(\mu)}\int_0^\infty t^{\mu-1}\left(\sum_{m\geq 1} a_m \big(x\mathrm{e}^{-t}\big)^{\lambda_m}\right)\left(\sum_{n\geq 1} b_n \big(x\mathrm{e}^{-t}\big)^{\rho_n}\right)\mathrm{d}t \\ &\leq \frac{\|\mathbf{a}\|_p\|\mathbf{b}\|_q}{\Gamma(\mu)}\int_0^\infty t^{\mu-1}\left(\sum_{m\geq 1}\big(x\mathrm{e}^{-t}\big)^{\lambda_m p'}\right)^{1/p'}\left(\sum_{n\geq 1}\big(x\mathrm{e}^{-t}\big)^{\rho_n q'}\right)^{1/q'}\mathrm{d}t\,. \end{aligned} \tag{17}$$

By the Cahen formula applied to (17) we get

$$\mathscr{D}_\lambda(p't) = \sum_{m\geq 1} x^{\lambda_m p'} \mathrm{e}^{-\lambda_m(p't)} = p't \int_0^\infty \mathrm{e}^{-p'tu} \sum_{m:\ \lambda_m \leq u} x^{\lambda_m p'} \, \mathrm{d}u$$

$$= p't \int_{\lambda_1}^\infty \mathrm{e}^{-p'tu} \sum_{m=1}^{[\lambda^{-1}(u)]} x^{\lambda_m p'} \, \mathrm{d}u \,.$$

Introducing the quasi-polynomial

$$Q_N^\varphi(z) = z^{\varphi_1} + \cdots + z^{\varphi_N}$$

we have

$$\mathscr{D}_\lambda(p't) = p't \int_{\lambda_1}^\infty \mathrm{e}^{-p'tu}\, Q^\lambda_{[\lambda^{-1}(u)]}\big(x^{p'}\big) \, \mathrm{d}u \,,$$

and analogously

$$\mathscr{D}_\rho(q't) = q't \int_{\rho_1}^\infty \mathrm{e}^{-q'tv}\, Q^\rho_{[\rho^{-1}(v)]}\big(x^{q'}\big) \, \mathrm{d}v \,.$$

Theorem 10

Let $p, q > 1$; p', q' be non–conjugated Hölder exponents to p, q respectively; $\mu > 0$, $\mathbf{a} = (a_n)_{n\in\mathbb{N}} \in \ell_p$, $\mathbf{b} = (b_n)_{n\in\mathbb{N}} \in \ell_q$ are nonnegative sequences and λ, ρ are positive monotone increasing functions satisfying (9). *Then*

$$\mathfrak{H}^{\mathbf{a},\mathbf{b}}_{\boldsymbol{\lambda},\boldsymbol{\rho}}(\mu; x) \le \mathsf{C}^{\mu}_{p',q'}(\lambda, \rho; x)\, \|\mathbf{a}\|_p \|\mathbf{b}\|_q\,, \tag{18}$$

where the constant

$$\mathsf{C}^{\mu}_{p',q'}(\lambda, \rho; x) = \frac{p'^{1/p'}\, q'^{1/q'}}{\Gamma(\mu)} \int_0^\infty t^{\mu+\Delta} \left(\int_{\lambda_1}^\infty \mathrm{e}^{-p'tu}\, Q^{\lambda}_{[\lambda^{-1}(u)]}(x^{p'})\, \mathrm{d}u \right)^{1/p'} \times \left(\int_{\rho_1}^\infty \mathrm{e}^{-q'tv}\, Q^{\rho}_{[\rho^{-1}(v)]}(x^{q'})\, \mathrm{d}v \right)^{1/q'} \mathrm{d}t\,.$$

The equality in (18) *appears for $\lambda(x) = \rho(x) = \mathcal{I}(x) = x$ and conjugated Hölder exponents $p = q = 2$ when*

$$a_m\, b_n^{-1} = C\, \delta_{mn} \qquad (m, n \in \mathbb{N};\ C > 0; q, x = 1).$$

Consequences of Theorem 10

F. An advantage of inequality class described in Theorem is the fact that Hölder exponents p, q are *non–conjugated and independent*. For the sake of simplicity taking $p = q$ and $p' = q'$ we arrive at the following result.

Corollary 8

Let all other assumption of Theorem 10 *valid. Then we get the following sharp ineqaulity:*

$$\mathfrak{H}_{\boldsymbol{\lambda},\boldsymbol{\rho}}^{\mathbf{a},\mathbf{b}}(\mu; x) \le \mathsf{C}_{p'}^{\mu}(\lambda, \rho; x)\, \|\mathbf{a}\|_p \|\mathbf{b}\|_p\,,$$

where the constant $\mathsf{C}_{p'}^{\mu}(\lambda, \rho; x)$ *becomes*

$$\frac{p'^{2/p'}}{\Gamma(\mu)} \int_0^\infty t^{\mu+\Delta} \left(\int_{\lambda_1}^\infty \int_{\rho_1}^\infty \mathrm{e}^{-p' t(u+v)}\, Q_{[\lambda^{-1}(u)]}^{\lambda}\big(x^{p'}\big)\, Q_{[\rho^{-1}(v)]}^{\rho}\big(x^{p'}\big)\, \mathrm{d}u\, \mathrm{d}v \right)^{1/p'} \mathrm{d}t\,.$$

G. Choosing $p' = q' = 2$ one deduces the following consequence of Theorem, such that is of considerable interest.

Corollary 9

Let $p' = q' = 2$ be conjugated Hölder exponents and let other assumptions of Theorem 10 *hold. For $\lambda = \rho$ we have the sharp inequality:*

$$\mathfrak{H}^{\mathbf{a},\mathbf{b}}_{\boldsymbol{\lambda},\boldsymbol{\lambda}}(\mu; x) \leq \mathsf{C}^{\mu}_{2,2}(\lambda; x)\, \|\mathbf{a}\|_p \|\mathbf{b}\|_q\,,$$

where

$$\mathsf{C}^{\mu}_{2,2}(\lambda; x) = \frac{\mu}{2^{\mu}} \int_{\lambda_1}^{\infty} \frac{Q^{\lambda}_{[\lambda^{-1}(u)]}\big(x^2\big)}{u^{\mu+1}}\, \mathrm{d}u\,.$$

REMARK. First, here p, q remain independent; we can only say that their range is $p, q \in (1, 2]$. Second, the equality of the kernel sequences $\boldsymbol{\lambda} = \boldsymbol{\rho}$ does not mean that the kernel of the Dirichlet–series $\mathfrak{H}^{\mathbf{a},\mathbf{b}}_{\boldsymbol{\lambda},\boldsymbol{\lambda}}(\mu; x)$ is homogeneous.

Finally, note that Corollary 9 cannot be deduced from Corollary 8 by any specification.

The results of this section are presented in the article

- B. Draščić Ban, J. Pečarić, I. Perić and T. Pogány, Discrete multiple Hilbert type inequality with non–homogeneous kernel, *J. Korean Math. Soc.* **47** (2010), No. 3, 537–546.

Let us define more general Hilbert type series. Let $m \in \mathbb{N}_2 := \{2, 3, \ldots\}$, $\mathbf{a}_j$, $j = \overline{1, m}$ be nonnegative sequences and let $\lambda_1, \ldots, \lambda_m \colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ be monotone increasing positive functions such that

$$\lim_{x \to \infty} \lambda_j(x) = \infty \qquad \big(j = \overline{1, m}\big), \tag{19}$$

and their restrictions are $\lambda_j\big|_{\mathbb{N}} = \boldsymbol{\lambda}_j = (\lambda_j(n_j))_{n_j=1}^{\infty}$. Let us denote $\mathbf{n} := (n_1, \ldots, n_m)$ an m–dimensional positive integer index (running on $\mathbb{N}^m$). Then the *multiple Hilbert–type series* reads

$$\mathfrak{H}_m(\rho, \mathbf{a}, \boldsymbol{\lambda}) := \sum_{\mathbf{n} \in \mathbb{N}^m} \frac{\prod_{j=1}^m a_{n_j}}{(\lambda_1(n_1) + \cdots + \lambda_m(n_m) + \rho)^{\mu}} \qquad \big(\rho, \mu > 0\big)\,. \tag{20}$$

Our main goal is to derive a sharp upper bound to (20).

Theorem 11 (Draščić Ban *et al.* (2010))

Assume $m \in \mathbb{N}_2$, $p_1^{-1} + \cdots + p_m^{-1} + q^{-1} \geq 1$, $\mu > 0$, $\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_m)$, Φ_i, $j = \overline{1, m}$ *positive monotone increasing functions and let* $\mathbf{a}_j$, $j = \overline{1, m}$ *be nonnegative sequences such that*

$$\big(\Phi_j^{1/q}(n_j) a_{n_j}\big)_{n_j=1}^{\infty} \in \ell_{p_j} \qquad \big(j = \overline{1, m}\big)$$

while $\boldsymbol{\lambda} = (\lambda_1, \ldots \lambda_m)$ *and all* λ_j *satisfies* (9). *Then we have*

$$\mathfrak{H}_m(\rho, \mathbf{a}, \boldsymbol{\lambda}) \leq \mathrm{C}_{\mu,q}^{\mathbf{\Phi}}(\mathbf{a}, \boldsymbol{\lambda}) \|\mathbf{a}_1 \mathbf{\Phi}_1^{1/q}\|_{p_1} \cdots \|\mathbf{a}_m \mathbf{\Phi}_m^{1/q}\|_{p_m}\,, \tag{21}$$

where the constant factor equals

$$\mathrm{C}_{\mu,q}^{\mathbf{\Phi}}(\mathbf{a}, \boldsymbol{\lambda}) = \left(\binom{\mu q + 1}{2} \int_{\boldsymbol{\lambda}(\mathbf{1})}^{\infty} \int_0^{[\boldsymbol{\lambda}^{-1}(\mathbf{t})]} \frac{\prod_{j=1}^m \mathfrak{d}_{u_j}(1/\Phi_j(u_j))}{(t_1 + \cdots + t_m + \rho)^{\mu q + 2}} \, \mathbf{dtdu} \right)^{1/q}. \tag{22}$$

Here $\int_{\boldsymbol{\lambda}(\mathbf{1})}^{\infty} := \int_{\lambda_1(1)}^{\infty} \cdots \int_{\lambda_m(1)}^{\infty}$; $\int_0^{[\boldsymbol{\lambda}^{-1}(\mathbf{t})]} := \int_0^{[\lambda_1^{-1}(t_1)]} \cdots \int_0^{[\lambda_m^{-1}(t_m)]}$ *stand as the suitable abbreviations for m–tuple integrals, while* $\mathbf{dx} := \mathrm{d}x_1 \cdots \mathrm{d}x_m$.

Hint for the proof.

Assume $q > 1$ and rewrite the m–tiple sum into the form:

$$\mathfrak{H}_m(\rho, \mathbf{a}, \boldsymbol{\lambda}) = \sum_{\mathbf{n} \in \mathbb{N}^m} \frac{\Phi_1^{1/q}(n_1) \cdots \Phi_m^{1/q}(n_m)\, a_{n_1} \cdots a_{n_m}}{\Phi_1^{1/q}(n_1) \cdots \Phi_m^{1/q}(n_m)(\lambda_1(n_1) + \cdots + \lambda_m(n_m) + \rho)^{\mu}}.$$

Then, making use of the generalized Hölder inequality with $p_1^{-1} + \cdots + p_m^{-1} + q^{-1} \geq 1$ [13], [18], we get

$$\mathfrak{H}_m(\rho, \mathbf{a}, \boldsymbol{\lambda}) \leq \Big(\sum_{n_1 \in \mathbb{N}} a_{n_1}^{p_1} \Phi_1^{p_1/q}(n_1) \Big)^{1/p_1} \cdots \Big(\sum_{n_m \in \mathbb{N}} a_{n_r} \Phi_m^{p_m/q}(n_m) \Big)^{1/p_m}$$
$$\times \left(\sum_{\mathbf{n} \in \mathbb{N}^m} \frac{1}{\Phi_1(n_1) \cdots \Phi_m(n_m)(\lambda_1(n_1) + \cdots + \lambda_m(n_m) + \rho)^{\mu q}} \right)^{1/q}.$$

Transform the general term of the multiple series by the Gamma function, we conclude after some algebra

$$\mathfrak{H}_m(\rho, \mathbf{a}, \boldsymbol{\lambda}) \leq \frac{\|\mathbf{a}_1 \boldsymbol{\Phi}_1^{1/q}\|_{p_1} \cdots \|\mathbf{a}_m \boldsymbol{\Phi}_m^{1/q}\|_{p_m}}{\Gamma^{1/q}(\mu q)}$$
$$\times \Bigg(\int_{\boldsymbol{\lambda}(\mathbf{1})}^{\infty} \int_0^{[\boldsymbol{\lambda}^{-1}(\mathbf{t})]} \mathfrak{d}_{u_1}\Big(\tfrac{1}{\Phi_1(u_1)}\Big) \cdots \mathfrak{d}_{u_m}\Big(\tfrac{1}{\Phi_m(u_m)}\Big)$$
$$\times \Bigg(\int_0^{\infty} e^{-(t_1+\cdots+t_m+\rho)x} x^{\mu q+1} \mathrm{d}x \Bigg) \mathrm{d}t_1 \cdots \mathrm{d}t_m \, \mathrm{d}u_1 \cdots \mathrm{d}u_m \Bigg)^{1/q} .$$

Now, the right hand bound becomes

$$R = \frac{\Gamma^{1/q}(\mu q + 2)}{\Gamma^{1/q}(\mu q)} \|\mathbf{a}_1 \boldsymbol{\Phi}_1^{1/q}\|_{p_1} \cdots \|\mathbf{a}_m \boldsymbol{\Phi}_m^{1/q}\|_{p_m}$$
$$\times \left(\int_{\boldsymbol{\lambda}(\mathbf{1})}^{\infty} \int_0^{[\boldsymbol{\lambda}^{-1}(\mathbf{t})]} \frac{\prod_{j=1}^m \mathfrak{d}_{u_j}\big(1/\Phi_j(u_j)\big)}{(t_1 + \cdots + t_m + \rho)^{\mu q+2}} \, \mathbf{dt} \, \mathbf{du} \right)^{1/q} . \qquad (23)$$

Since all above involved series are convergent, the associated integral expressions are convergent as well. So, all interchanges of integration order are enabled. □

Applications to Mordell–Tornheim–Witten Zeta function

In the same article

- B. Draščić Ban, J. Pečarić, I. Perić and T. Pogány, Discrete multiple Hilbert type inequality with non–homogeneous kernel, *J. Korean Math. Soc.* **47** (2010), No. 3, 537–546.

certain integral expressions have been derived for the so–called *Mordell–Tornheim Zeta function*. In 1950 Tornheim [28] introduced the double series

$$T(p,q,r) = \sum_{n_1,n_2 \geq 0} \frac{1}{n_1^p n_2^q (n_1+n_2)^r} \qquad \big(p,q,r>0,\ p+\min\{q,r\}>1\big)\,. \tag{24}$$

In honour to the author it is called *Tornheim double series*, also known in literature as *Witten Zeta function* or *Mordell series*.
Later, a various continuations of the domain for the function $T(p,q,r)$ are given by J. M. Borwein (2008), Espinosa and Moll (2006), Matsumoto (2003), Subbarao and Sitaramachandra Rao (1985), Tsumura (2007) and others, but here we are interested in the case $p>1, q>1, r>0$.

Theorem 12 (Draščić Ban *et al.* (2010))

Let $p > 1, q > 1, r > 0$. Then the series $T(p, q, r)$ possesses the integral representation:

$$T(p,q,r) = \frac{1}{4\Gamma(p)\Gamma(q)\Gamma(r)} \int_0^\infty \int_0^\infty \int_0^\infty \frac{x^{r-1} t_1^{p-1} t_2^{q-1}\, e^{-x-\frac{t_1+t_2}{2}}}{\sinh \frac{x+t_1}{2}\, \sinh \frac{x+t_2}{2}} \, \mathrm{d}x \, \mathrm{d}t_1 \, \mathrm{d}t_2 \,. \tag{25}$$

Theorem 13 (Draščić Ban *et al.* (2010))

Let $p > 1, q > 1, r > 0$. Then we have

$$T(p,q,r) = \binom{r+1}{2} \int_1^\infty \int_1^\infty \int_0^{[t_1]} \int_0^{[t_2]} \frac{\mathfrak{d}_{u_1}(u_1^{-p})\mathfrak{d}_{u_2}(u_2^{-q})}{(t_1+t_2)^{r+2}} \, \mathrm{d}t_1 \mathrm{d}t_2 \mathrm{d}u_1 \mathrm{d}u_2 \,, \tag{26}$$

where

$$\mathfrak{d}_x^{\mathfrak{q}} = \mathfrak{q} + \{\mathfrak{q}x\} \frac{\partial}{\partial x} \,.$$

Applying the Final remark 2 to the Theorem 9:

$$\sum_{n\geq 1} a_n \mathrm{e}^{-\lambda_n x} = \sum_{n\geq 1} \frac{a_n}{\Phi(n)} \cdot \Phi(n) \mathrm{e}^{-\lambda_n x}$$

extended to double series case, we arrive at

Theorem 14

Suppose $p_1^{-1} + p_2^{-1} + q^{-1} \geq 1$, $\mu > 0$ *and* $\mathbf{a}_j \geq 0$ *are such, that* $\big(\Phi_j^{1/q}(n_j) a_{n_j}\big) \in \ell_{p_j}$, $j = 1, 2$. *Then*

$$\sum_{n_1, n_2 \geq 1} \frac{a_{n_1} a_{n_2}}{(\lambda_1(n_1) + \lambda_2(n_2))^\mu} \leq \mathrm{C}_{\mu,q}^{\boldsymbol{\Phi}}(\mathbf{a}, \boldsymbol{\lambda}) \|\mathbf{a}_1 \boldsymbol{\Phi}_1^{1/q}\|_{p_1} \|\mathbf{a}_2 \boldsymbol{\Phi}_2^{1/q}\|_{p_2}\,.$$

Here the constant $\mathrm{C}_{\mu,q}^{\boldsymbol{\Phi}}(\mathbf{a}, \boldsymbol{\lambda})$ *takes the value*

$$\left(\binom{\mu q + 1}{2} \int_{\lambda_1(1)}^\infty \int_{\lambda_2(1)}^\infty \frac{\mathrm{d}t_1 \mathrm{d}t_2}{(t_1 + t_2)^{\mu q + 2}} \right.$$

$$\left. \times \int_0^{[\lambda_1^{-1}(t_1)]} \int_0^{[\lambda_2^{-1}(t_2)]} \mathfrak{d}_{u_1}(1/\Phi_1(u_1)) \mathfrak{d}_{u_1}(1/\Phi_1(u_1))\, \mathrm{d}u_1 \mathrm{d}u_2 \right)^{1/q}.$$

We end the exposition with connecting Mathieu series, Dirichlet series, Hilbert's double series theorem and Mordell–Tornheim Zeta functions integral representation, by choosing $\Phi_j(n_j) = n_j^{r_j}$, $j = 1, 2$; $\lambda_j = \mathcal{I}$.

Theorem 15

Suppose $p_1^{-1} + p_2^{-1} + q^{-1} \geq 1$, $\mu > 0$, $\mathbf{a}_j \geq 0$ *are such that*

$$\left((n_j a_{n_j})^{r_j/q}\right) \in \ell_{p_j} \qquad j = 1, 2$$

and λ_1, λ_2 *are positive monotone increasing to the infinity. Then*

$$\sum_{n_1, n_2 = 1}^{\infty} \frac{a_{n_1} a_{n_2}}{(n_1 + n_2)^{\mu}} \leq \mathrm{C}_{\mu,q}^{\mathbb{N}^{\mathbf{r}}}(\mathbf{a}, \mathcal{I}) \|\mathbf{a}_1 \mathbf{n}_1^{r_1/q}\|_{p_1} \|\mathbf{a}_2 \mathbf{n}_2^{r_2/q}\|_{p_2} \,,$$

where the inequality's constant factor $\mathrm{C}_{\mu,q}^{\mathbb{N}^{\mathbf{r}}}(\mathbf{a}, \mathcal{I})$ *equals*

$$\left(\binom{\mu q + 1}{2} \int_1^{\infty} \int_1^{\infty} \int_0^{[t_1]} \int_0^{[t_2]} \frac{\mathfrak{d}_{u_1}(u_1^{-p_1}) \mathfrak{d}_{u_2}(u_2^{-p_2})}{(t_1 + t_2)^{\mu q + 2}} \mathrm{d}t_1 \mathrm{d}t_2 \mathrm{d}u_1 \mathrm{d}u_2 \right)^{1/q} .$$

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