Fredholm property and essential spectrum of $3-D$ Dirac operators with regular and singular potentials

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Introduction

The Schrödinger operators with $\delta-$potentials supported on surfaces in $\mathbb{R}^n$ play an important role in Mathematical Physics and have attracted a lot of attention in the recent past; they are used for the description of quantum particles interacting with charged hypersurfaces, in an approximation of the Hamiltonian of propagation of electrons through a thin barriers, and in many other models of motion of particles.

The mathematical problems connected with formal Schrödinger operators with singular potentials

$$\mathbb{S} = -\Delta + a\delta_\Sigma$$

where Σ is the set of Lebesgue measure 0 in $\mathbb{R}^n$ include the realization of the formal Schrödinger operator $\mathbb{S}$ as an unbounded operator $\mathcal{S}$ in the Hilbert space $L^2(\mathbb{R}^n)$ and the study of the spectral properties of $\mathcal{S}$.

Over the past two decades, this topic has been intensively studied and there is extensive literature devoted to this problem.

The aim of the talk is the spectral properties of $3-D$ Dirac operators with singular potentials supported on bounded and unbounded surfaces in $\mathbb{R}^3$. The $3-D$ Dirac operators are the basic operators of relativistic quantum mechanics and quantum field theory and their spectral properties are important from both theoretical and applied point of view.
See the books:

- Bjorken, J.D., Drell,S.D.: Relativistic Quantum Mechanics, McGraw-Hill Book Company, New York St. Louis San Francisco Toronto London Sydney (1964)
- Thaller, B.: The Dirac Equation, Springer-Verlag, Berlin Heidelberg NewYork London (1956).
- Bogolubov, N.N., Shirkov; D.V., Quantum Fields, Benjamin/Cummings Publishing Company Inc. (1982).

The abstract Dirac operators is an important tool of the index theory and the spectral geometry. See:

- Gilkey,P. B., Invariance theory, the heat equation, and the Atiyah-Singer index theorem, seconded., CRC Press, Boca Raton, FL, 199

The investigation of the $3D-$Dirac operators with singular potentials supported on compact closed $\cap$surfaces in $\mathbb{R}^3$ was initiated only recently in the pioneering paper:

- N. Arrizabalaga, A. Mas, and L. Vega. Shell interactions for Dirac operators. J. Math. Pures Appl. (9), 102(4):617–639, 2014

where a new approach to extension theory of symmetric operators was employed;
This research was continued in the papers:

- Ourmieres-Bonafos, Th.,Vega,L.: A strategy for self-adjointness of Dirac operators: Applications to the MIT BAG model and shell interactions. Publ.Mat. 62, 397-437,(2018).
- Moroianu, A.,Ourmíeres-Bonafos,Th.,Pankrashkin, K.: Dirac operators on surfaces large mass limits, Journal Math. Pures et Appliquees, V. 102, Is. 4, Pages 617 - 639 (2014).
- Ourmieros-Bonafos, Th., Pizzichlllo,F., Dirac operators and shell interactions: a survey, arXiv:1902.03901v1 [math-ph] 11 Feb 2019.094 (2015).

A different approach using the abstract theory of quasi-boundary triples and their Weyl functions was proposed in:

- Behrndt,J.,Exner,P.,Holzmann,M.,Lotoreichik,V.: On the spectral properties of Dirac operators with electrostatic δ-shell interactions, J. Math.Pures Appl.111, 47–78, (2018).
- Behrndt,J.,Exner,P.,Holzmann,M.,Lotoreichik,V: On Dirac operators in $\mathbb{R}^3$ with electrostatic and Lorentz scalar δ-shell interactions, Quantum Stud.: Math. Found., https://doi.org/10.1007/s40509-019-00186-6, (2019).

In contrast to the indicated papers, we consider singular potentials with supports on both bounded and unbounded surfaces in $\mathbb{R}^3$. Our approach to the self-adjointness of Dirac operators is based on the study of the interaction (transmission) problems with parameter associated with the Dirac operators. We introduce the Lopatinsky conditions for their invertibility for large values of the parameter, and for the a priori estimates of solutions of associated interaction (transmission) problems.

Moreover we study the Fredholm properties and the essential spectrum of unbounded operators associated with the Dirac operators with singular potentials with supports on compact surfaces and non-compact surfaces with conical exits to infinity. For this aim we apply the limit operators method, see for instance:

- Rabinovich,V.S.,Roch,S.,Silbermann,B.: Limit Operators and their Applications in Operator Theory, In ser.Operator Theory: Advances and Applications, vol 150, Birkhäuser Verlag, (2004).
- Rabinovich,V.S.: Essential spectrum of perturbed pseudodifferential operators. Applications to the Schrödinger, Klein-Gordon, and Dirac operators, Russ. J. Math. Physics, 12:1, 62-80, (2005).
- Rabinovich,V.S.: Transmission problems for conical and quasi-conical at infinity domains, Applicable Analysis, Vol. 94, No. 10, 2077–2094 (2015).

Description of the problem

We consider the formal Dirac operators with singular potentials

$$D_{\mathbf{A},\Phi,Q_s}\mathbf{u}(x) = \left(\mathfrak{D}_{\mathbf{A},\Phi} + Q_{\text{sin}}\right)\mathbf{u}(x), x \in \mathbb{R}^3 \tag{1}$$

where

$$\mathfrak{D}_{\mathbf{A},\Phi} = \sum_{j=1}^{3} \alpha_j \left(i\partial_{x_j} + A_j\right) + \alpha_0 m + \Phi I_4 \tag{2}$$

is the Dirac operators defined on vector-valued distributions $\mathbf{u}$ on $\mathbb{R}^3$ with values in $\mathbb{C}^4$.

In formula (2)

$$\alpha_0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, j = 1, 2, 3, \tag{3}$$

are 4×4 Dirac matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

are the 2×2 Pauli matrices, for which the Clifford relations hold

$$\alpha_j \alpha_k + \alpha_k \alpha_j = \delta_{jk} I_4, j, k = 0, 1, 2, 3, \tag{5}$$

$$\sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{jk} I_2, j, k = 1, 2, 3 \tag{6}$$

where I_n is the $n \times n$ unit matrix.

- In the equation (2) $\mathbf{A} = (A_1, A_2, A_3)$ is the vector-valued potential of the magnetic field $\mathbf{H} = \boldsymbol{\nabla} \times \mathbf{A}$, Φ is the electrostatic potential of the electric field $\mathbf{E} = \nabla\Phi$, $m \in \mathbb{R}$ is the mass of the particle. We use the system of coordinates for which the Planck constant $\hbar = 1$, the light speed $c = 1$, and the charge of the particle $e = 1$. In what follows we assume that $A_j, \Phi \in L^\infty(\mathbb{R}^3)$, and A_j are real-valued functions.
- The singular potential $Q_{\sin}$ is of the form $Q_{\sin} = \Gamma\delta_S$ where $\Gamma = (\Gamma_{ij})_{i,j=1}^4$ is a 4×4 matrix, $\Gamma_{ij} \in C_b(S)$ the space of contnuous bounded functions on S, and δ_S is the delta-function with support on an C^2-surface $S \subset \mathbb{R}^3$ which divides $\mathbb{R}^3$ on two open domains $\Omega_\pm$ with the common boundary S. The distribution $\Gamma\delta_S$ acts on the test functions $\boldsymbol{\varphi} \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$ as

$$\Gamma\delta_S(\boldsymbol{\varphi}) = \int_S \Gamma(s)\boldsymbol{\varphi}(s)ds,$$

where ds is the Lebesgue surface measure on S, We denote $L^2(\mathbb{R}^3, \mathbb{C}^4)$ the Hilbert space of 4-dimensional vector-functions $\mathbf{u}(x) = (u^1(x), u^2(x), u^3(x), u^4(x)), x \in \mathbb{R}^3$ with the scalar product

$$\langle \mathbf{u}, \mathbf{v} \rangle = \int_{\mathbb{R}^3} (\mathbf{u}(x), \mathbf{v}(x))_{\mathbb{C}^4} \, dx,$$

$H^s(\mathbb{R}^3, \mathbb{C}^4)$ is the Sobolev space of vector-functions on $\mathbb{R}^3$ with values in $\mathbb{C}^4, H^s(\Omega_\pm, \mathbb{C}^4)$ are the spaces of restrictions of functions in $H^s(\mathbb{R}^3, \mathbb{C}^4)$ on domains $\Omega_\pm$.

Let $\Omega_{\pm}$ be unbounded domains in $\mathbb{R}^3$ and S be a C^2- surface being their common boundary.
We assume that S is *uniformly regular,* that is:
(i) There exists $r > 0$ such that for every point $x_0 \in S$ there exists the diffeomorphism $\varphi_{x_0} : B_r(x_0) \to B_1(0)$ where $B_r(x_0) = \left\{x \in \mathbb{R}^3 : |x - x_0| < r\right\}$ such that

$$\varphi_{x_0}\left(B_r(x_0) \cap \Omega_{\pm}\right) = B_1(0) \cap \mathbb{R}^3_{\pm}, \mathbb{R}^3_{\pm} = \left\{y = (y', y_3) \in \mathbb{R}^3 : y_3 \gtrless 0\right\},$$
$$\varphi_{x_0}\left(B_r(x_0) \cap \Sigma\right) = B_1(0) \cap \mathbb{R}^2_{y'}, \mathbb{R}^2_{y'} = \left\{y = (y', y_3) \in \mathbb{R}^3 : y_3 = 0\right\}.$$

(ii) Let $\psi_{x_0} = \varphi_{x_0}^{-1}$ and $\varphi^i_{x_0}, \psi^i_{x_0}$ be the coordinate functions of the mappings $\varphi_{x_0}, \psi_{x_0}$. Then

$$\sup_{x_0 \in S} \sup_{|\alpha| \le 2, x \in B_r(x_0)} \left|\partial^\alpha \varphi^i_{x_0}(x)\right| < \infty, \sup_{x_0 \in S} \sup_{|\alpha| \le 2, y \in B_1(0)} \left|\partial^\alpha \psi^i_{x_0}(y)\right| < \infty,$$
$$i = 1, 2, 3.$$

Realization of Dirac operators with singular potentials as unbounded operators

Let $H^1(\Omega_\pm, \mathbb{C}^4)$ be the spaces of restrictions on $\Omega_\pm$ of distributions in $H^1(\mathbb{R}^3, \mathbb{C}^4)$, and let $H^1(\mathbb{R}^3 \diagdown S, \mathbb{C}^4) = H^1(\Omega_+, \mathbb{C}^4) \oplus H^1(\Omega_-, \mathbb{C}^4)$. We set $\mathbf{u}_\pm = \gamma_S^\pm \mathbf{u}$ where $\gamma_S^\pm : H^1(\Omega_\pm, \mathbb{C}^4) \to H^{1/2}(S, \mathbb{C}^4)$ are the trace operators.

The product of the singular potential $Q_s = \Gamma \delta_S$ and the discontinuous function $\mathbf{u} \in H^1(\mathbb{R}^3 \diagdown S, \mathbb{C}^4)$ is defined as the distribution $\Gamma \delta_S \mathbf{u} \in \mathcal{D}'(\mathbb{R}^3, \mathbb{C}^4) = \mathcal{D}'(\mathbb{R}^3) \otimes \mathbb{C}^4$ acting on $\boldsymbol{\varphi} \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$ as

$$(\Gamma \delta_S \mathbf{u})(\boldsymbol{\varphi}) = \frac{1}{2} \int_S \left(\Gamma(s) \left(\mathbf{u}_+(s) + \mathbf{u}_-(s) \right), \boldsymbol{\varphi}(s) \right)_{\mathbb{C}^4} ds. \tag{7}$$

Let $\mathbf{u} \in H^1(\mathbb{R}^3 \diagdown S)$. Then in the sense of distributions in $\mathcal{D}'(\mathbb{R}^3, \mathbb{C}^4)$

$$D_{\mathbf{A},\Phi,Q_{\sin}}\mathbf{u} = \tag{8}$$

$$\{\mathfrak{D}_{\mathbf{A},\Phi}\mathbf{u}\} - \left[i\boldsymbol{\alpha} \cdot \boldsymbol{\nu}\left(\mathbf{u}_+ - \mathbf{u}_-\right) - \frac{1}{2}\Gamma\left(\mathbf{u}_+ + \mathbf{u}_-\right)\right]\delta_S,$$

where

$$\boldsymbol{\alpha}\cdot\boldsymbol{\nu} = \nu_1\alpha_1 + \nu_2\alpha_2 + \nu_3\alpha_3,$$

$\boldsymbol{\nu}(\mathbf{s}) = (\nu_1(s), \nu_2(s), \nu_3(s))$, $s \in S$ is the normal outward to Ω_+ vector. We denote by $\{\mathfrak{D}_{\mathbf{A},\Phi}\mathbf{u}\}$ the regular distribution which coinsides with $\mathfrak{D}_{\mathbf{A},\Phi}\mathbf{u}^{\pm}(x)$,for $x \in \Omega_{\pm}$, where $\mathbf{u}^{\pm} = \mathbf{u}\,|_{\Omega_{\pm}}$.

Thus $D_{\mathbf{A},\Phi,Q_{\sin}}\mathbf{u} \in L^2(\mathbb{R}^3, \mathbb{C}^4)$ if and only if the following interaction (transmission) condition on the surfaces S holds

$$i\boldsymbol{\alpha}\cdot\boldsymbol{\nu}(s)\left(\mathbf{u}_+(s) - \mathbf{u}_-(s)\right) - \frac{1}{2}\Gamma(s)\left(\mathbf{u}_+(s) + \mathbf{u}_-(s)\right) = \mathbf{0}, s \in S. \tag{9}$$

Condition (9) can be written of the form

$$a_+(s)\mathbf{u}_+(s) + a_-(s)\mathbf{u}_-(s) = 0, s \in S \tag{10}$$

where

$$a_+(s) = \frac{1}{2}\Gamma(s) - i\boldsymbol{\alpha}\cdot\boldsymbol{\nu}(s), a_-(s) = \frac{1}{2}\Gamma(s) + i\boldsymbol{\alpha}\cdot\boldsymbol{\nu}(s) \tag{11}$$

are 4×4 matrix-function.

We associate with the formal Dirac operator $D_{\mathbf{A},\Phi,Q_{\sin}}$ the unbounded in $L^2(\mathbb{R}^3, \mathbb{C}^4)$ operator $\mathcal{D}_{\mathbf{A},\Phi,a_+,a_-}$ defined by the regular Dirac operator

$$\mathfrak{D}_{\mathbf{A},\Phi} = \boldsymbol{\alpha}\boldsymbol{\cdot}(\mathbf{D}+\mathbf{A}) + \alpha_0 m + \Phi I_4, \mathbf{D} = (i\partial_{x_1}, i\partial_{x_2}, i\partial_{x_3})$$

with domain

$$dom\mathcal{D}_{\mathbf{A},\Phi,,a_+,a_-} = H^1_{a_+,a_-}(\mathbb{R}^3\diagdown S, \mathbb{C}^4) =$$
$$\{u \in H^1(\mathbb{R}^3\diagdown S, \mathbb{C}^4) : a_+(s)\mathbf{u}_+(s) + a_-(s)\mathbf{u}_-(s) = \mathbf{0}, s \in S$$

We associate with $D_{\mathbf{A},\Phi,Q_{\text{sin}}}$ also the bounded operator of the transmission problem

$$\mathbb{D}_{\mathbf{A},\Phi,a_+,a_-}u(x) = \begin{cases} \mathfrak{D}_{\mathbf{A},\Phi}\mathbf{u}(x), x \in \mathbb{R}^3 \diagdown S \\ a_+(s)\mathbf{u}_+(s) + a_-(s)\mathbf{u}_-(s) = \mathbf{0}, s \in S \end{cases} \tag{12}$$

acting from $H^1(\mathbb{R}^3 \diagdown S, \mathbb{C}^4)$ into $L^2(\mathbb{R}^3, \mathbb{C}^4)$.

Parameter dependent Dirac operators with singular potentials

In what follows we also need the parameter dependent operator of the transmission problem

$$\mathbb{D}_{\mathbf{A},\Phi,a_+,a_-}(\mu)\mathbf{u}(x) = \left(\mathbb{D}_{\mathbf{A},\Phi,a_+,a_-} + i\mu I_4\right)\mathbf{u}(x)$$
$$= \begin{cases} (\mathfrak{D}_{\mathbf{A},\Phi} + i\mu I_4)\mathbf{u}(x), x \in \mathbb{R}^3 \diagdown S, \\ a_+(s)\mathbf{u}_+(s) + a_-(s)\mathbf{u}_-(s) = 0, s \in S \end{cases}, \mu \in \mathbb{R} \tag{13}$$

acting from $H^1(\mathbb{R}^3 \diagdown S, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)$.

The equality

$$(\boldsymbol{\alpha} \cdot \mathbf{D} - i\mu I_4)(\boldsymbol{\alpha} \cdot \mathbf{D} + i\mu I_4) = \left(-\Delta + \mu^2\right) I_4 \tag{14}$$

yields that $\mathfrak{D}_{\mathbf{A},\Phi} + i\mu I_4$ is the elliptic parameter-dependent operator.

We consider the invertibility of

$$\mathbb{D}_{\mathbf{A},\Phi,a_+,a_-}(\mu) : H^1(\mathbb{R}^3 \diagdown S, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)$$

for large value of the parameter μ following to the well-known Agranovich-Vishik paper

- Agranovich MS,Vishik MI. Elliptic problems with a parameter and parabolic problems of general forms. Uspekhi Mat. Nauk. 1964,219; 63–161; English trans. Russian Math. Surveys. 1964; 19; 53–157.

We obtain the local parameter-dependent estimates and then we glue them using the admissible partition of unity.
The local estimates at the point $x_0 \in \mathbb{R}^3 / S$ are based on the uniform ellipticity of $\mathfrak{D}_{\mathbf{A},\Phi}$.
The local estimates at the point $x_0 \in S$ we use the local system of orthogonal coordinates $y = (y_1, y_2, y_3)$ where the axis y_1, y_2 belong to the tangent plane to S at the point $x_0 \in S$ and the axis $y_3 = z$ is directs along the outward to Ω_+ normal vector $\boldsymbol{\nu}$ to S at the point $x_0 \in S$. Passing to this local system of coordinates and taking the main part of $\mathfrak{D}_{\mathbf{A},\Phi}$ we obtain the following parameter-depending transmission problem for half-spaces $\mathbb{R}^3_\pm = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_3 \lessgtr 0\}$:

$$\mathfrak{B}_{a_+(x_0),a_-(x_0)}(\mu)\boldsymbol{\psi}(x) =$$
$$\begin{cases} \left(\boldsymbol{\alpha}' \cdot \mathbf{D}_{y'} + i\alpha_3 \frac{\partial}{\partial y_3} + i\mu I_4\right) \boldsymbol{\psi}(y) = 0, y \in \mathbb{R}^3_+ \cup \mathbb{R}^3_- \\ a_+(x_0)\boldsymbol{\psi}_+(\mathbf{y}', 0) + a_-(x_0)\boldsymbol{\psi}_-(\mathbf{y}', 0) = 0, y' \in \mathbb{R}^2 \end{cases}$$
$$a_\pm(x_0) = \frac{1}{2}\Gamma(x_0) \mp i\alpha_3.$$

Parameter-dependent Lopatinsky conditions

We consider the invertibility of the operator

$$\mathfrak{B}_{a_+(x_0),a_-(x_0)}(\mu) : H^1(\mathbb{R}^3_+,\mathbb{C}^4) \oplus H^1(\mathbb{R}^3_-,\mathbb{C}^4) \to L^2(\mathbb{R}^3,\mathbb{C}^4).$$

After the Fourier transform with respect to $y' = (y_1, y_2) \in \mathbb{R}^2$ we obtain the 1−dimensional transmission problem depending on the parameters $(\boldsymbol{\xi}', \mu) \in \mathbb{R}^2 \times \mathbb{R}$

$$\hat{\mathfrak{B}}_{a_+(x_0),a_-(x_0)}(\boldsymbol{\xi}',\mu)\psi(\boldsymbol{\xi}',\mu,z) = \begin{cases} \left(\boldsymbol{\alpha}'\cdot\boldsymbol{\xi}' + i\alpha_3\frac{d}{dz} + i\mu I_4\right)\boldsymbol{\psi}(\boldsymbol{\xi}',\mu,z), z \in \mathbb{R}\diagdown\{0\}, \\ a_+(x_0)\boldsymbol{\psi}_+(\boldsymbol{\xi}',\mu,0) + a_-(x_0)\boldsymbol{\psi}_-(\boldsymbol{\xi}',\mu,0) = \mathbf{0}. \end{cases},$$

$$y_3 = z, \boldsymbol{\xi}' = (\xi_1,\xi_2) \in \mathbb{R}^2, \boldsymbol{\alpha}'\cdot\boldsymbol{\xi}' = \alpha_1\xi_1 + \alpha_2\xi_2$$

The operator $\hat{\mathfrak{B}}_{a_+(x_0),a_-(x_0)}(\boldsymbol{\xi}',\mu)$ acts from $H^1(\mathbb{R}_+,\mathbb{C}^4) \oplus H^1(\mathbb{R}_-,\mathbb{C}^4)$ into $L^2(\mathbb{R},\mathbb{C}^4)$.

One can prove that the operator

$$\mathfrak{B}_{a_+(x_0),a_-(x_0)}(\mu) : H^1(\mathbb{R}^3_+, \mathbb{C}^4) \oplus H^1(\mathbb{R}^3_-, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)$$

is invertible for all $\mu \in \mathbb{R}$ if and only if the homogeneous equation

$$\hat{\mathfrak{B}}_{a_+(x_0),a_-(x_0)}(\boldsymbol{\xi}', \mu)\boldsymbol{\psi} = \mathbf{0}$$

has the trivial solution in $L^2(\mathbb{R}, \mathbb{C}^4)$, only for all $(\boldsymbol{\xi}', \mu) \in \mathbb{R}^2 \times \mathbb{R} : |\boldsymbol{\xi}'|^2 + \mu^2 = 1$.

The homogeneous equation

$$\left(\boldsymbol{\alpha}' \cdot \boldsymbol{\xi}' + i\alpha_3 \frac{d}{dz} + i\mu I_4\right) \boldsymbol{\psi} = \mathbf{0} \tag{15}$$

has the exponential solutions

$$\boldsymbol{\psi}_{\pm}(z) = \mathbf{h}^{\pm} e^{\pm\rho z}, \rho = \sqrt{\left|\boldsymbol{\xi}'\right|^2 + \mu^2} \tag{16}$$

where $\mathbf{h}^{\pm} \in \mathbb{C}^4$. Substituting (16) in equation (15) we obtain that the vectors $\mathbf{h}^{\pm} \in \mathbb{C}^4$ satisfy the matrix equation

$$\Lambda_{\pm}(\boldsymbol{\xi}', \mu)\mathbf{h}^{\pm} = \left(\boldsymbol{\alpha} \cdot \boldsymbol{\xi}' \pm i\rho\alpha_3 + i\mu I_4\right) \mathbf{h}^{\pm} = \mathbf{0}. \tag{17}$$

The four vectors

$$\mathbf{h}_{1,+}(\boldsymbol{\xi}',\mu)=\begin{pmatrix} i\mu\mathbf{e}_1 \\ \Lambda_+(\boldsymbol{\xi}',\mu)e_1 \end{pmatrix},\mathbf{h}_{2,+}(\boldsymbol{\xi}',\mu)=\begin{pmatrix} \Lambda_+(\boldsymbol{\xi}',\mu)\mathbf{e}_2 \\ i\mu\mathbf{e}_2 \end{pmatrix}, \quad (18)$$

$$\mathbf{h}_{1,-}(\boldsymbol{\xi}',\mu)=\begin{pmatrix} i\mu\mathbf{e}_1 \\ \Lambda_-(\boldsymbol{\xi}',\mu)\mathbf{e}_1 \end{pmatrix},\mathbf{h}_{2,-}(\boldsymbol{\xi}',\mu)=\begin{pmatrix} \Lambda_-(\boldsymbol{\xi}',\mu)\mathbf{e}_2 \\ i\mu\mathbf{e}_2 \end{pmatrix},$$

$$\mathbf{e}_1=\begin{pmatrix} 1 \\ 0 \end{pmatrix},\mathbf{e}_2=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

form the orthogonal base of linear space of solutions of equation (17). Hence the fundamental system of solutions of the differential equation

$$\left(\boldsymbol{\alpha}'\cdot\boldsymbol{\xi}'+i\alpha_3\frac{d}{dz}+i\mu I_4\right)\boldsymbol{\psi}=\mathbf{0}$$

is

$$\mathbf{h}_{1,\pm}(\boldsymbol{\xi}',\mu)e^{\pm\rho z},\mathbf{h}_{2,\pm}(\boldsymbol{\xi}',\mu)e^{\pm\rho z}.$$

The exponentially decreasing solutions of equation (15) are of the form

$$\boldsymbol{\psi}(z,\boldsymbol{\xi}',\mu)=\begin{array}{l}\left(C_1^+\mathbf{h}_{+,1}(\boldsymbol{\xi}',\mu)+C_2^+\mathbf{h}_{+,2}(\boldsymbol{\xi}',\mu)\right)e^{\rho z},z<0\\ \left(C_1^-\mathbf{h}_{-,1}(\boldsymbol{\xi}',\mu)+C_2^-\mathbf{h}_{-,2}(\boldsymbol{\xi}',\mu)\right)e^{-\rho z},z>0\end{array},\qquad(19)$$

where $C_1^+, C_2^+, C_1^-, C_2^-$ are arbitrary constants. Substituting $\boldsymbol{\psi}(z,\boldsymbol{\xi}',\mu)$ in the interaction conditions

$$a_+(x_0)\psi_+(\boldsymbol{\xi}',\mu,0)+a_-(x_0)\psi_-(\boldsymbol{\xi}',\mu,0)=0, s\in S.$$

we obtain the 4×4 system of linear equations

$$C_1^+a_-(x_0)\mathbf{h}_{+,1}+C_2^+a_-(x_0)\mathbf{h}_{+,2}+C_1^-a_+(x_0)\mathbf{h}_{-,1}+C_2^-a_+(x_0)\mathbf{h}_{+,2}=0 \qquad(20)$$

with respect to $(C_1^+, C_2^+, C_1^-, C_2^-)$.

System (20) has the trivial solution if and only if

$$\det \mathcal{M}(x_0, \boldsymbol{\xi}', \mu) \neq 0 \text{ for every } (\xi', \mu) : \ \mu^2 + \left|\boldsymbol{\xi}'\right|^2 = 1, \tag{21}$$

where $\mathcal{M}(x_0, \boldsymbol{\xi}', \mu)$ is the matrix with columns

$$\left\{ a_-(x_0)\mathbf{h}_{+,1}(\boldsymbol{\xi}', \mu), a_-(x_0)\mathbf{h}_{+,2}(\boldsymbol{\xi}', \mu), a_+(x_0)\mathbf{h}_{-,1}(\boldsymbol{\xi}', \mu), a_+(x_0)\mathbf{h}_{-,1}(\boldsymbol{\xi}', \mu) \right\}$$

We say that the **parameter-dependent Lopatinsky condition** holds at the point $x_0 \in S$ if the condition (21) is satisfied, and we say that the **uniform parameter-dependent Lopatinsky condition** holds if

$$\inf_{x_0 \in S, \mu^2 + \left|\boldsymbol{\xi}'\right|^2 = 1} \left|\det \mathcal{M}(x_0, \boldsymbol{\xi}', \mu)\right| > 0. \tag{22}$$

Theorem

Let $S \subset \mathbb{R}^3$ be a C^2- closed compact surface or C^2-unbounded uniformly regular surface, $\mathbf{A} = (A_1, A_2, A_3) \in L^\infty(\mathbb{R}^3, \mathbb{C}^4)$, $\Phi \in L^\infty(\mathbb{R})$, $\Gamma \in C_b(S, \mathcal{B}(\mathbb{C}^4))$, and the uniformly Lopatinsky conditions (22) for parameter-dependent operator $\mathfrak{D}_{\mathbf{A},\Phi,a_+,a_-}(\mu), \mu \in \mathbb{R}$ are satisfied. Then there exists $\mu_0 > 0$ such that the operator

$$\mathbb{D}_{\mathbf{A},\Phi,a_+,a_-}(\mu) : H^1(\mathbb{R}^3 \diagdown S, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)$$

is invertible for every $\mu \in \mathbb{R} : |\mu| > \mu_0$.

Self-adjointness of the operators $\mathcal{D}_{\mathbf{A},\Phi,a_+,a_-}$

Theorem

Let $S \subset \mathbb{R}^3$ be an either compact C^2-surface or uniformly regular C^2-surface, the potentials $\mathbf{A} \in L^\infty(\mathbb{R}^3, \mathbb{R}^4), \Phi \in L^\infty(\mathbb{R}^3)$ be real-valued, and $\Gamma = (\Gamma_{ij})_{i,j=1}^4$ be the Hermitian matrix with elements $\Gamma_{ij} \in C_b(S)$. Let the parameter-dependent Lopatinsky condition (22) for the operator $\mathbb{D}_{\mathbf{A},\Phi,a_+,a_-}(\mu)$ be satisfied uniformly. Then the operator $\mathcal{D}_{\mathbf{A},\Phi,a_+,a_-}$ is self-adjoint in $L^2(\mathbb{R}^3, \mathbb{C}^4)$.

Proof.

Let $\mathbf{u}, \mathbf{v} \in H^1_{a_+,a_-}(\mathbb{R}^3 \diagdown S, \mathbb{C}^4) = Dom\ \mathcal{D}_{\mathbf{A},\Phi,a_+,a_-}$. Then integrating by parts and taking into account that $\mathbf{u}, \mathbf{v} \in H^1_{a_+,a_-}(\mathbb{R}^3 \diagdown S, \mathbb{C}^4)$ we obtain that

$$\langle \mathfrak{D}_{\mathbf{A},\Phi}\mathbf{u}, \mathbf{v}\rangle_{L^2(\mathbb{R}^3,\mathbb{C}^4)} = \langle \mathbf{u}, \mathfrak{D}_{\mathbf{A},\Phi}\mathbf{v}\rangle_{L^2(\mathbb{R}^3,\mathbb{C}^4)} \cdot$$

Hence $\mathcal{D}_{\mathbf{A},\Phi,a_+,a_-}$ is the symmetric operator. The previous theorem implies that the operator

$$\mathbb{D}_{\mathbf{A},\Phi,a_+,a_-}(\mu) : H^1_{a_+,a_-}(\mathbb{R}^3 \diagdown S, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)$$

is invertible for $|\mu|$ large enough. It implies that the unbounded operator $\mathcal{D}_{\mathbf{A},\Phi,a_+,a_-}$ is closed and the deficiency indices of $\mathcal{D}_{\mathbf{A},\Phi,a_+,a_-}$ are 0. Hence the operator $\mathcal{D}_{\mathbf{A},\Phi,a_+,a_-}$with domain $H^1_{a_+,a_-}(\mathbb{R}^3 \diagdown S, \mathbb{C}^4)$ is self-adjoint in $L^2(\mathbb{R}^3, \mathbb{C}^4)$. □

Electrostatic and Lorentz delta-shell interactions

As example we consider the $3D-$Dirac operator

$$D_{\mathbf{A},\boldsymbol{\Phi},\mathbf{Q}_{\sin}} = , (\mathbf{D}+\mathbf{A}) + \alpha_0 m + \Phi I_4 + 2(\eta I_4 + \tau\alpha_0)\delta_S. \qquad (23)$$

We assume that $A_j, \Phi \in L^\infty(\mathbb{R}^3)$, and $\eta, \tau \in C_b(S)$, S is a closed compact C^2-surface or an unbounded C^2-uniformly regular surface. The matrices $a_\pm$ are of the form

$$a_\pm(s) = \eta(s)I_4 + \tau(s)\alpha_0 \mp i\boldsymbol{\alpha}\cdot\boldsymbol{\nu}(s).$$

We associate with the formal Dirac operator $D_{\mathbf{A},\boldsymbol{\Phi},\mathbf{Q}_{\sin}}$ the unbounded operator $\mathcal{D}_{\mathbf{A},\boldsymbol{\Phi},a_+,a_-}$ in $L^2(\mathbb{R}^3,\mathbb{C}^4)$ with domain $H^1_{a_+,a_-}(\mathbb{R}^3\diagdown S)$.

Theorem

Let $\mathbf{A} \in L^{\infty}(\mathbb{R}^3, \mathbb{C}^4)$, $\in L^{\infty}(\mathbb{R}^3), \eta, \tau \in C_b(S)$ *be real-valued, and*

$$\inf_{s \in S} \left| \eta^2(s) - \tau^2(s) - 1 \right| > 0 \tag{24}$$

Then the operator $\mathcal{D}_{\mathbf{A},\mathbf{\Phi},a_+,a_-}$ *is self-adjoint.*

Proof.

Theorem 3 follows from Theorem 2 since condition (24) yields the parameter-dependent uniform Lopatinsky condition for the associated transmission problem. □

Remark. In the recent paper

- Behrndt,J.,Exner,P.,Holzmann,M.,Lotoreichik,V: On Dirac operators in $\mathbb{R}^3$ with electrostatic and Lorentz scalar δ-shell interactions, Quantum Stud.: Math. Found., https://doi.org/10.1007/s40509-019-00186-6, (2019)

the autors consider the singular potential $2\left(\eta I_4 + \tau\alpha_0\right)\delta_S$ for constant $\eta, \tau \in \mathbb{R}$ and a compact closed C^2 interaction surface S. They proved the self-adjointness $\mathcal{D}_{\mathbf{0},0,a_+,a_-}$ with zero regular potentials for the condition $\eta^2 - \tau^2 \neq 1$. Moreover the authors considered the degenerated case $\eta^2 - \tau^2 = 1$. They show that in the degenerated case the transmission problem is reduced to two independent so-called **MIT-Bag boundary problems** for domains $\Omega_\pm$.

Fredholm property and Essential Spectrum

We consider the Fredholm property of the transmission operator

$$\mathbb{D}_{\mathbf{A},\Phi,a_+,a_-}\mathbf{u}(x) = \begin{cases} \mathfrak{D}_{\mathbf{A},\Phi}\mathbf{u}(x), x \in \mathbb{R}^3 \diagdown S \\ a_+(s)\mathbf{u}_+(s) + a_-(s)\mathbf{u}_-(s) = \mathbf{0}, s \in S; \end{cases} \quad (25)$$

associated with the formal Dirac operator with singular potentials $\mathfrak{D}_{\mathbf{A},\Phi,Q_{\sin}} = \mathfrak{D}_{\mathbf{A},\Phi} + Q_{\sin}$ where

$$\mathfrak{D}_{\mathbf{A},\Phi}u(x) = \boldsymbol{\alpha} \cdot (\mathbf{D}_x + \mathbf{A}(x)) + \alpha_0 m + \Phi(x)\mathbf{u}(x), x \in \mathbb{R}^3 \diagdown S \quad (26)$$

and

$$Q_{\sin} = \Gamma\delta_S.$$

We assume that S is a connected C^2-surface being the common boundary of the domain $\Omega_\pm$. We consider the case if either S is a compact closed C^2-surface or surface with the conical structure at infinity which is automatically uniformly regular.

Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \leq \chi(x) \leq 1, \chi(x) = 1$ for $|x| \leq 1$, and $\chi(x) = 0$ for $|x| \geq 2$, and $\chi_R(x) = \chi(\frac{x}{R})$, $\psi_R(x) = 1 - \chi_R(x)$.

- We say that the operator $\mathfrak{D}_{\mathbf{A},\Phi,a_+,a_-}$ is **locally Fredholm if for every** $R > 0$ there exist operators

$$\mathcal{L}_R, \mathcal{R}_R \in \mathcal{B}(L^2(\mathbb{R}^3, \mathbb{C}^4), H^1(\mathbb{R}^3 \diagdown S, \mathbb{C}^4))$$

such that

$$\mathcal{L}_R \mathfrak{D}_{\mathbf{A},\Phi,a_+,a_-} \chi_R I = \chi_R I + T_R', \chi_R \mathfrak{D}_{\mathbf{A},\Phi,a_+,a_-} \mathcal{R}_R = \chi_R I + T_R'', \quad (27)$$

where $T_R' \in \mathcal{K}\left(H^1(\mathbb{R}^3 \diagdown S, \mathbb{C}^4)\right)$, $T_R'' \in \mathcal{K}\left(L^2(\mathbb{R}^3, \mathbb{C}^4)\right)$;

- We say that the operator $\mathfrak{D}_{\mathbf{A},\Phi,a_+,a_-}$ is **locally invertible at infinity if there exists** $R > 0$ and operators $\mathcal{L}_R^\infty, \mathcal{R}_R^\infty \in \mathcal{B}(L^2(\mathbb{R}^3, \mathbb{C}^4), H^1(\mathbb{R}^3 \diagdown S, \mathbb{C}^4))$ such that

$$\mathcal{L}_R^\infty \mathfrak{D}_{\mathbf{A},\Phi,a_+,a_-} \psi_R I = \psi_R I, \psi_R \mathfrak{D}_{\mathbf{A},\Phi,a_+,a_-} \mathcal{R}_R^\infty = \psi_R I. \quad (28)$$

Lemma

The operator $\mathfrak{D}_{\mathbf{A},\Phi,a_+,a_-} : H^1(\mathbb{R}^3 \diagdown S, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)$ is Fredholm if and only if $\mathfrak{D}_{\mathbf{A},\Phi,a_+,a_-}$ is locally Fredholm and locally invertible at infinity.

Compact interaction surfaces

First we consider the Fredholm property of the operator $\mathfrak{D}_{\mathbf{A},\Phi,a_+,a_-}$ if S is a **compact closed** C^2-**surface**.

Definition

We say that $a \in UC_{b,\infty}(\mathbb{R}^n)$ if $a \in L^\infty(\mathbb{R}^n)$ and there exists $R > 0$ such that $a_R = \psi_R a \in UC_b(\mathbb{R}^n)$ where ψ_R is the above defined cut-off function.

Let $a \in UC_{b,\infty}(\mathbb{R}^n)$. We consider the functional sequence

$$b_m(x) = a_R(x + g_m), \mathbb{R} \ni g_m \to \infty.$$

This sequence is uniformly bounded and equicontinuous. By the Arcela-Ascoli Lemma there exists a subsequence h_m of g_m and the limit function $a_R^h \in C_b(\mathbb{R}^n)$ such that

$$\lim_{m\to\infty} \sup_{x\in K} \left| a_R(x + h_m) - a_R^h(x) \right| = 0 \qquad (29)$$

for every compact set $K \subset \mathbb{R}^n$. Note that the function $a_R^h = a^h$ is independent of R.

Definition

We assume that the vector-valued potential $\mathbf{A} = (A_1, A_2, A_3)$ and the scalar potentials Φ are such that

$$A_j, \Phi \in UC_{b,\infty}(\mathbb{R}^n), \tag{30}$$

$$\Gamma(s) = (\Gamma_{ij}(s))_{i,j=1}^{4}, \Gamma_{ij} \in C_b(S). \tag{31}$$

Let $g_m \to \infty$. Then there exists a subsequence h_m of g_m and limit functions $A_j^h(x), j = 1, 2, 3, \Phi^h(x)$ in $C_b(\mathbb{R}^n)$. The operator

$$\mathfrak{D}^h_{\mathbf{A},\Phi} = \mathfrak{D}_{\mathbf{A}^h,\Phi^h}$$

is called the limit operator of $\mathfrak{D}_{\mathbf{A},\Phi}$. We denote by $Lim(\mathfrak{D}_{\mathbf{A},\Phi})$ the set of all limit operators of $\mathfrak{D}_{\mathbf{A},\Phi}$.

Theorem

Let $A_j, \Phi \in UC_{b,\infty}(\mathbb{R}^n)$, S be the compact closed surface being the common boundary Ω_+ and Ω_-, and the **Lopatinsky conditions** *be satisfied at every point $s \in S$, that is*

$$\det \mathcal{M}(s, \boldsymbol{\xi}', 0) \neq 0 \text{ for every } \xi' : |\xi'| = 1.$$

Then

$$\mathbb{D}_{\mathbf{A},\Phi,a_+,a_-} : H^1(\mathbb{R}^3 \diagup S, \mathbb{C}^4) \to L^2\left(\mathbb{R}^3, \mathbb{C}^4\right)$$

is the Fredholm operator if and only if all limit operators $\mathfrak{D}^h_{\mathbf{A},\Phi} \in Lim(\mathfrak{D}_{\mathbf{A},\Phi})$ are invertible from $H^1(\mathbb{R}, \mathbb{C}^4)$ into $L^2\left(\mathbb{R}^3, \mathbb{C}^4\right)$.

Proof.

The ellipticity of $\mathfrak{D}_{\mathbf{A},\Phi}$ and the Lopatinsky condition provide the local Fredholmness of $\mathbb{D}_{\mathbf{A},\Phi,a_+,a_-}$. Hence $\mathbb{D}_{\mathbf{A},\Phi,a_+,a_-}$ is the Fredholm operator if and only if $\mathbb{D}_{\mathbf{A},\Phi,a_+,a_-}$ is locally invertible at infinity. But $\mathbb{D}_{\mathbf{A},\Phi,a_+,a_-}$ coinsides with $\mathfrak{D}_{\mathbf{A},\Phi}$ at infinity. Invertibility of $\mathfrak{D}_{\mathbf{A},\Phi}$ is equivalent to invertibiliy all limit operators $\mathfrak{D}_{\mathbf{A}^h,\Phi^h}$. □

Corollary

Let $A_j, \Phi \in UC_{b,\infty}(\mathbb{R}^n)$, S *be a* C^2-*compact surface,* $\mathcal{D}_{\mathbf{A}^h,\Phi^h}$ *be an unbounded operator in* $L^2(\mathbb{R}^3, \mathbb{C}^4)$ *defined by the Dirac operators* $\mathfrak{D}_{\mathbf{A}^h,\Phi^h}$ *with domain* $H^1(\mathbb{R}^3, \mathbb{C}^4)$. *Moreover let* **the Lopatinsky condition** *be satisfied at every point* $x \in S$. *Then*

$$sp_{ess}\mathcal{D}_{\mathbf{A},\Phi,a_+,a_-} = \bigcup_h sp\mathcal{D}_{\mathbf{A}^h,\Phi^h}$$

where the union is taken with respect to all sequences tending to infinity and defining the limit functions $\mathbf{A}^h, \Phi^h$.

Slowly oscillating data

Definition

We say that a function $f \in L^\infty(\mathbb{R}^n)$ is slowly oscillating at infinity if

$$\lim_{x\to\infty} \sup_{y\in K} |f(x+y) - f(x)| = 0$$

for every compact set $K \subset \mathbb{R}^n$. We denote the class of slowly oscillating at infinity functions by $SO_\infty(\mathbb{R}^n)$.

Simple example: $\sin |x|^\alpha, 0 < \alpha < 1$ is slowly oscillating at infinity. Note that $SO_\infty(\mathbb{R}^n) \subset UC_{b,\infty}(\mathbb{R}^n)$. Moreover, if $f \in SO_\infty(\mathbb{R}^n)$ and $\mathbb{R}^n \ni h_m \to \infty$ is such that there exists a limit function f^h. **Then the function f^h is a constant.**

Compact surface of interactions

Assume that the interaction surface S is compact, $\Gamma_{i,j} \in C(S), i,j = 1,...,4$, the vector and scalar potentials $\mathbf{A} = (A_1, A_2, A_3)$, and Φ are real-valued and slowly oscillating at infinity. Then the limit vector potentials $\mathbf{A}^h \in \mathbb{R}^3$ and limit scalar potentials $\Phi^h \in \mathbb{R}$. For the calculation of the essential spectra we need the limit operators $\mathfrak{D}_{\mathbf{A}^h,\Phi^h}$ for $\mathfrak{D}_{\mathbf{A},\Phi}$:

$$\mathfrak{D}_{\mathbf{A}^h,\Phi^h} u(x) = \left(\boldsymbol{\alpha} \cdot \left(\mathbf{D} + \mathbf{A}^h\right) + \alpha_0 m + \Phi^h\right) u(x), x \in \mathbb{R}^3 \quad (32)$$

where $A_j^h, \Phi^h \in \mathbb{R}$.

Note that

$$sp\mathfrak{D}_{\mathbf{A}^h,\Phi^h} = sp\left(\boldsymbol{\alpha}\cdot\left(\mathbf{D}+\mathbf{A}^h\right)+\alpha_0 m+\Phi^h\right) = sp\left(\boldsymbol{\alpha}\cdot\mathbf{D}+\alpha_0 m+\Phi^h\right)$$
$$= \left(-\infty,-|m|+\Phi^h\right]\cup[|m|+\Phi^h,+\infty).$$

Hence if the Lopatinsky conditions are satisfied at every point $s\in S$ then

$$sp_{ess}\mathcal{D}_{\mathbf{A},\Phi,,a_+,a_-} = \bigcup sp\mathfrak{D}_{\mathbf{A}^h,\Phi^h} = (-\infty, M_\Phi^{\sup}-|m|]\bigcup[M_\Phi^{\inf}+|m|,+\infty) \tag{33}$$

where

$$M_\Phi^{\sup} = \limsup_{x\to\infty}\Phi(x),\ M_\Phi^{\inf} = \liminf_{x\to\infty}\Phi(x).$$

That is the $sp_{ess}\mathcal{D}_{\mathbf{A},\Phi,,a_+,a_-}$ is independent on the magnetic potential $\mathbf{A}$ and the singular potential $Q_{\sin}$.

As example we consider the $3D-$Dirac operator

$$D_{\mathbf{A},\mathbf{\Phi},\mathbf{Q}_{\text{sin}}} = , (\mathbf{D}+\mathbf{A}) + \alpha_0 m + \Phi I_4 + 2(\eta I_4 + \tau\alpha_0)\delta_S \tag{34}$$

with delta-shell Lorentz and electrostatic potentials, where A_j,$\Phi \in SO_\infty(\mathbb{R}^3)$ are real-valued functions, and

$$\eta^2(s) - \tau^2(s) \neq 1, s \in S.$$

Then $\mathcal{D}_{\mathbf{a},\Phi,a_+,a_-}$ is the self-adjoint operator and $sp_{ess}\mathcal{D}_{\mathbf{A},\Phi,,a_+,a_-}$ is defined by formula (33).

Interaction on unbounded surfaces with conic exits at infinity

Let the unbounded C^2-surface S has the conic structure at infinity. That is there exists $R>0$ such that if $x_0 \in S : |x_0| > R$, then the ray

$$\left\{x \in \mathbb{R}^3 : x = tx_0, t > 0\right\} \subset S.$$

We denote by $\widetilde{\mathbb{R}^n}$ the compactification of $\mathbb{R}^n$ obtained by the adjoint to every ray

$$l_\omega = \left\{x \in \mathbb{R}^n : x = t\omega, t > 0, \omega \in S^2\right\}.$$

the infinitely distant point ϑ_ω. The topology in $\widetilde{\mathbb{R}^n}$ is introduced such that $\widetilde{\mathbb{R}^n}$ is homeomorphic to the closed unit ball $\bar{B}_1(0)$. We denote by $\widetilde{\Omega_\pm}, \tilde{S}$ the compactifications of $\Omega_\pm, S$ in the topology of $\widetilde{\mathbb{R}^3}$.

We consider the operator of interaction problem

$$\mathbb{D}_{\mathbf{A},\Phi,a_+,a_-}u(x) = \begin{cases} \mathfrak{D}_{\mathbf{A},\Phi}\mathbf{u}(x), x \in \mathbb{R}^3 \diagdown S, \\ a_+(s)\mathbf{u}_+(s) + a_-(s)\mathbf{u}_-(s) = 0, s \in S \end{cases},$$

$$a_+(s) = \frac{1}{2}\Gamma(s) - i\alpha \cdot \nu(s), a_-(s) = \frac{1}{2}\Gamma(s) + i\alpha \cdot \nu(s), s \in S$$

We assume that

$$A_j, j = 1, 2, 3, \Phi \in SO^\infty(\mathbb{R}^3), a_\pm(s) \in C(\tilde{S}). \tag{35}$$

Limit operators

(i) Let $\mathbb{R}^3 \ni h_m \to \vartheta_\omega \notin S_\infty$, and $A_j^h, j = 1, 2, 3, \Phi^h$ are the limit functions defined by the sequence $h = (h_m)$.

Then the limit operator $\mathfrak{D}^h_{\mathbf{A},\Phi,a_+,a_-}$ defined by the sequence $h = (h_m)$ is

$$\mathfrak{D}^h_{\mathbf{A},\Phi} = \mathfrak{D}_{\mathbf{A}^h,\Phi^h} : H^1(\mathbb{R}^3, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4).$$

(ii) Let $\mathbb{R}^3 \ni h_m \to \vartheta_\omega \in S_\infty = \tilde{S} \diagdown S$. Then the limit functions are such that:

$$A_j^h \in \mathbb{R}, j = 1, 2, 3, \Phi^h \in \mathbb{C}, \Gamma_{i,j}^{\vartheta_\omega} \in \mathbb{R}.$$

Let $l_\omega^R = \left\{ x \in \mathbb{R}^3 : x = t\omega, t > R \right\}$ be tha ray and T_{ϑ_ω} be the tangent plane to S at this ray l_ω^R. We denote by $\mathbb{R}^3_{\pm,\vartheta_\omega}$ the half-spaces in $\mathbb{R}^3$ with the common boundary T_{ϑ_ω}. After the passing to suitable coordinate system as above we obtain the transmission problem

$$\mathbb{D}_{\mathbf{A}^h,\Phi^h,a_+^h,a_-^h} \mathbf{v}(y) = \left\{ \begin{array}{c} \mathfrak{D}_{\mathbf{A}^h,\Phi^h} \mathbf{v}(y), y \in \mathbb{R}^3 \diagdown \mathbb{R}^2, \\ a_+^{\vartheta_\omega} \mathbf{v}_+(y', 0) + a_-^{\vartheta_\omega} \mathbf{v}_-(y', 0) = \mathbf{0}, y' \in \mathbb{R}^2 \end{array} \right. ,$$

$$a_+^{\vartheta_\omega} = \frac{1}{2}\Gamma^{\vartheta_\omega} - i\alpha_3, a_-^{\vartheta_\omega} = \frac{1}{2}\Gamma^{\vartheta_\omega} + i\alpha_3.$$

where

$$\mathfrak{D}_{\mathbf{A}^h,\Phi^h} = \boldsymbol{\alpha} \cdot (\mathbf{D} + \mathbf{A}^h) + \alpha_0 m + \Phi^h I_4,$$

$$\mathbf{A}^h \in \mathbb{R}^3, \Phi^h \in \mathbb{C}, \Gamma^{\vartheta_\omega} = \lim_{s \to \vartheta_\omega} \Gamma(s).$$

Theorem

Let $A_j, j = 1,2,3, \Phi \in SO^{\infty}(\mathbb{R}^3)$, $\Gamma_{ij} \in C(\tilde{S}), i,j = 1,...,4$, *and the* **Lopatinsky conditions** *be satisfied at every point* $s \in S$. *Then the operator*

$$\mathfrak{D}_{\mathbf{A},\Phi,a_+,a_-} : H^1(\mathbb{R}^3 \diagdown S, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)$$

is Fredholm if and only if:
(i) for every $\vartheta_\omega \notin S_\infty$ *all limit operators*

$$\mathfrak{D}^h_{\mathbf{A},\Phi} : H^1(\mathbb{R}^3, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)$$

defined by the sequences $h_m \to \vartheta_\omega$ *are invertible;*
(ii) for every $\vartheta_\omega \in S_\infty$ *all limit operators*

$$\mathfrak{D}_{\mathbf{0},\Phi^h,a_+^{\vartheta_\omega},a_-^{\vartheta_\omega}} : H^1(\mathbb{R}^3 \diagdown \mathbb{R}^2, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)$$

defined by the sequences $h_m \to \vartheta_\omega$ *are invertible.*

Theorem

Let $\mathcal{D}_{\mathbf{A},\Phi,a_+,a_-}$ *be an unbounded operator in* $L^2(\mathbb{R}^3,\mathbb{C}^4)$ *associated with the operator* $\mathbb{D}_{\mathbf{A},\Phi,a_+,a_-}$ *of interaction problem, and* $\mathcal{D}^h_{\mathbf{A},\Phi,a_+,a_-}$ *are unbounded operators associated with* $\mathfrak{D}^h_{\mathbf{A},\Phi,a_+,a_-}$. *Then*

$$sp_{ess}\mathcal{D}_{\mathbf{A},\Phi,a_+,a_-} = \bigcup_{\mathcal{D}^h_{\mathbf{0},\Phi,a_+,a_-}\in Lim\mathcal{D}_{\mathbf{A},\Phi,a_+,a_-}} sp\mathcal{D}^h_{\mathbf{0},\Phi,a_+,a_-} \qquad (36)$$

where $Lim\mathcal{D}_{\mathbf{A},\Phi,a_+,a_-}$ *is the set of all limit operators of* $\mathcal{D}_{\mathbf{A},\Phi,a_+,a_-}$.

Let us consider $\mathfrak{D}_{\mathbf{0},\Phi^h,a_+^{\vartheta_\omega},a_-^{\vartheta_\omega}}$ for $h_m \to \vartheta_\omega$. Applying the Fourier transform we obtain a family of $1-D$ transmission problem

$$\hat{\mathfrak{D}}_{\mathbf{0},0,a_+^{\vartheta_\omega},a_-^{\vartheta_\omega}}(\xi')\boldsymbol{\varphi}(z) = \begin{cases} \left(\boldsymbol{\alpha}' \cdot \boldsymbol{\xi}' + \alpha_0 m + i\alpha_3 \frac{d}{dz}\right)\boldsymbol{\varphi}(z), z \in \mathbb{R} \diagdown \{0\} \\ a_+^{\vartheta_\omega}\boldsymbol{\varphi}_+(0) + a_-^{\vartheta_\omega}\boldsymbol{\varphi}_-(0) = \mathbf{0}. \end{cases} \tag{37}$$

$$a_+^{\vartheta_\omega} = \frac{1}{2}\Gamma^{\vartheta_\omega} - i\alpha_3, a_-^{\vartheta_\omega} = \frac{1}{2}\Gamma^{\vartheta_\omega} + i\alpha_3. \tag{38}$$

The unbounded operator $\hat{\mathfrak{D}}_{\mathbf{0},0,a_+^{\vartheta\omega},a_-^{\vartheta\omega}}(\xi')$ has the essential spectrum

$$sp_{ess}\hat{\mathfrak{D}}_{\mathbf{0},0,a_+^{\vartheta\omega},a_-^{\vartheta\omega}}(\boldsymbol{\xi}') = \left(-\infty, -\sqrt{\left|\boldsymbol{\xi}'\right|^2 + m^2}\right] \bigcup \left[\sqrt{\left|\boldsymbol{\xi}'\right|^2 + m^2}, +\infty\right)$$

and perhaps a finite set of points of the discrete spectrum on the interval $\left(-\sqrt{\left|\boldsymbol{\xi}'\right|^2 + m^2}, \sqrt{\left|\boldsymbol{\xi}'\right|^2 + m^2}\right)$. Hence

$$\begin{aligned} sp\mathfrak{D}_{\mathbf{0},0,a_+^{\vartheta\omega},a_-^{\vartheta\omega}} &= \bigcup_{\xi'\in\mathbb{R}^2} sp_{ess}\hat{\mathfrak{D}}_{\mathbf{0},0,a_+^{\vartheta\omega},a_-^{\vartheta\omega}}(\boldsymbol{\xi}') \bigcup_{\xi'\in\mathbb{R}^2} sp_{dis}\mathfrak{D}_{\mathbf{0},0,a_+^{\vartheta\omega},a_-^{\vartheta\omega}}(\xi') \\ &= (-\infty, -|m|] \bigcup [|m|, +\infty) \\ &\quad \bigcup_{\xi'\in\mathbb{R}^2} sp_{dis}\mathfrak{D}_{\mathbf{0},0,a_+^{\vartheta\omega},a_-^{\vartheta\omega}}(\xi') \cap (-|m|, |m|). \end{aligned}$$

If for all $\vartheta_\omega \in S_\infty$

$$sp_{dis}\mathfrak{D}_{\mathbf{0},0,a_+^{\vartheta_\omega},a_-^{\vartheta_\omega}}(\xi') \cap (-|m|,|m|) = \varnothing \text{ for all } \xi' \in \mathbb{R}^2.$$

Then

$$sp_{ess}\mathcal{D}_{\mathbf{A},\Phi,,a_+,a_-} = \bigcup sp\mathfrak{D}_{\mathbf{A}^h,\Phi^h} = (-\infty, M_\Phi^{\sup} - |m|]\bigcup[M_\Phi^{\inf} + |m|,+\infty) \tag{39}$$

where

$$M_\Phi^{\sup} = \limsup_{x\to\infty}\Phi(x), M_\Phi^{\inf} = \liminf_{x\to\infty}\Phi(x).$$

That is

$$sp_{ess}\mathcal{D}_{\mathbf{A},\Phi,,a_+,a_-} = sp_{ess}\mathcal{D}_{\mathbf{A},0}.$$