Grand Lebesgue space for $p = \infty$ and applications or a new life of a 36 years old result of Nikolai Karapetyants and Boris Rubin

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This presentation is based on the paper
Grand Lebesgue space for $p = \infty$ and its application to Sobolev-Adams embedding theorems in borderline cases
by H. Rafeiro, S. Samko, and S. Umarkhadzhiev

Grand Lebesgue spaces for finite p (GLS)

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Definition (Grand Lebesgue spaces)

$L^{p)}_{\theta}(\Omega)$, $1 < p < \infty$, $\theta > 0$, $\Omega \subset \mathbb{R}^n$, $|\Omega| < \infty$, are defined by the norm

$$\|f\|_{L^{p)}_{\theta}(\Omega)} := \sup_{0<\varepsilon<p-1} \varepsilon^{\theta} \left(\int_{\Omega} |f(x)|^{p-\varepsilon} \,\mathrm{d}x \right)^{\frac{1}{p-\varepsilon}} . \qquad (1)$$

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- Few names involved in the study of GLS: A. Fiorenza, V. Kokilashvili, A. Meskhi, J.M. Rakotoson, P. Jain, etc.

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Definition (Grand $L^{\infty)}_\psi$ spaces)

$L^{\infty)}_\psi(\Omega)$ may be realized via the norm

$$\|f\|_{L^{\infty)}_\psi(\Omega)} = \sup_{q>1} \frac{1}{\psi(q)} \|f\|_{L^q(\Omega)} \tag{2}$$

where

$$\inf_{q>1} \psi(q) > 0 \quad \text{and} \quad \lim_{q\to\infty} \psi(q) = \infty. \tag{3}$$

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- The proof was based on direct obtaining asymptotics of the $L^p \to L^q$ norms of one-dimensional fractional operator $\frac{1}{q} = \frac{1}{p} - \alpha$ as $p \to \frac{1}{\alpha}$, which essentially used one-dimensional techniques.

Grand Lebesgue spaces for $p = \infty$ (GLS-∞)

- From papers [1,2] of N. Karapetyants and B. Rubin, we have a hint that these GLS-∞ are useful in studying mapping properties of Riesz potential operator in the borderline case.

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- Our approach is based on estimations of the growth of constants as p tends to the borderline cases when using Hedberg approach

$$|I^\alpha_\Omega f(x)| \leqslant C_1 r^\alpha Mf(x) + C_2 r^{\alpha - \frac{n}{p}} \|f\|_{L^p(\Omega)} \quad r > 0,$$

Grand Lebesgue spaces for $p = \infty$ (GLS-∞)

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1. We show that such spaces are useful in the study of mapping properties of the Riesz potential operator in the borderline cases $\alpha p = n$ for Lebesgue spaces and $\alpha p = n - \lambda$ for Morrey and Morrey type spaces.

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2. Some statements on improving BMO results for Lebesgue spaces were known, as regards to Morrey and Morrey type spaces, there was no improving BMO results.

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1. We show that such spaces are useful in the study of mapping properties of the Riesz potential operator in the borderline cases $\alpha p = n$ for Lebesgue spaces and $\alpha p = n - \lambda$ for Morrey and Morrey type spaces.
2. Some statements on improving BMO results for Lebesgue spaces were known, as regards to Morrey and Morrey type spaces, there was no improving BMO results.
3. We also show that the obtained results are sharp in a certain sense.

Riesz potential operator

Definition (Riesz potential operator)

$$I^{\alpha}_{\Omega} f(x) := \int_{\Omega} \frac{f(y)}{|x-y|^{n-\alpha}}\, \mathrm{d}y, \ x \in \Omega,$$

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$L_\psi^{\infty)}$ spaces

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Example (Function in GLS-∞)

Let $\Omega = B(0, R)$, $R > 0$, and $f(x) = \left(\ln\left(\frac{R}{|x|}\right)\right)^{\gamma}$ with $\gamma > 0$. Then $f \in L_{\theta}^{\infty)}(B(0, R))$ if and only if $\gamma \leqslant \theta$.

Theorem (Pointwise estimate for the Riesz potential)

Let $\Omega \subseteq \mathbb{R}^n$ *be an open set,* $p = \frac{n}{\alpha}$, $f \in \bigcap_{0<\varepsilon<p-1} L^{p-\varepsilon}(\Omega)$, *and* $0 < \varepsilon < p - 1$. *Then*

$$|I_\Omega^\alpha f(x)| \leqslant C\left(r^\alpha Mf(x) + \frac{1}{\varepsilon^{\frac{1}{p'}}} r^{\alpha - \frac{n}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}(\Omega)}\right), \tag{5}$$

where C *does not depend on* f *and* ε.

Theorem (Estimate for the Riesz potential of a function)

Let $1 < p < \infty$, $\varepsilon > 0$, $1/p + \varepsilon < 1$, *and*

$$f(x) = \frac{1}{|x|^{\frac{n}{p}} \left(\ln \frac{C}{|x|}\right)^{\frac{1}{p}+\varepsilon}} \in L^p(B), \quad B = B(0,R), \quad C = R^2 \cdot \mathrm{e}^2.$$

Then

$$I_B^\alpha f(x) \simeq \left(\ln \frac{1}{|x|}\right)^{\frac{1}{p'}-\varepsilon} \quad \text{as} \quad x \to 0,$$

when $\alpha = \frac{n}{p}$.

Main result on Lebesgue spaces

Theorem (Lebesgue spaces)

Let $|\Omega| < \infty$, $0 < \alpha < n$, and $p = \frac{n}{\alpha}$. Then the operator I^α_Ω is bounded from the classical Lebesgue space $L^p(\Omega)$ to $L^{\infty)}_\theta(\Omega)$ with $\theta = \frac{1}{p'}$ and this choice of θ is sharp.

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Sharp in the sense that $I_\Omega^\alpha : L^p \to L_\theta^{\infty)}$, $\theta < \frac{1}{p'}$, is not true.

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Sharp in the sense that $I^\alpha_\Omega : L^p \to L^{\infty)}_\theta$, $\theta < \frac{1}{p'}$, is not true.

The operator I^α_Ω is also bounded from the grand space $L^{p)}_\theta(\Omega)$ to the space $L^{\infty)}_\xi(\Omega)$, where $\xi = \theta + \frac{1}{p'}$ and this choice of ξ is sharp.

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Theorem (Lebesgue spaces)

Under the assumptions of Theorem above, the operator I^α_Ω maps the Lebesgue space $L^p(\Omega)$ into $\mathrm{BMO}(\Omega) \cap L^{\infty)}_\theta(\Omega)$, $\theta = \frac{1}{p'}$.

Sketch of the Proof

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$$\varepsilon^{\frac{1}{p'}} \left\| I_\Omega^\alpha f \right\|_{L^{q_\varepsilon}(\Omega)} \leqslant C \|Mf\|_{L^{p-\varepsilon}(\Omega)}^{\frac{\varepsilon}{p}} \|f\|_{L^{p-\varepsilon}(\Omega)}^{1-\frac{\varepsilon}{p}}. \tag{6}$$
where $\frac{1}{q_\varepsilon} = \frac{1}{p-\varepsilon} - \frac{\alpha}{n} = \frac{\varepsilon}{p(p-\varepsilon)}$.

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- By the boundedness of M in $L^{p-\varepsilon}$ and some estimates, from (6) we obtain
$$\begin{aligned} \sup_{q>p'+\eta} \frac{1}{q^{\theta+\frac{1}{p'}}} \left\| I_\Omega^\alpha f \right\|_{L^q(\Omega)} &\leqslant C \sup_{0<\varepsilon<p-1-\delta} \varepsilon^\theta \| f \|_{L^{p-\varepsilon}(\Omega)} \\ &= C \| f \|_{L_\theta^{p)}(\Omega)} , \end{aligned}$$

Morrey spaces

Definition (Morrey spaces)

Let $1 \leqslant p < \infty$ and $0 \leqslant \lambda < n$. We recall that Morrey spaces $L^{p,\lambda}(\Omega)$ are defined as the set of measurable functions on Ω such that

$$\|f\|_{L^{p,\lambda}(\Omega)} := \sup_{r>0, x\in\Omega} \frac{1}{r^{\frac{\lambda}{p}}} \left(\int_{\Omega(x,r)} |f(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}} < \infty,$$

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Definition (grand Morrey spaces)

We recall that grand Morrey spaces $L_\theta^{p),\lambda}(\Omega)$ are defined as the set of measurable functions on Ω such that

$$\|f\|_{L_\theta^{p),\lambda}(\Omega)} := \sup_{0<\varepsilon<p-1} \varepsilon^\theta \|f\|_{L^{p-\varepsilon,\lambda}(\Omega)} < \infty. \tag{7}$$

Known results for mapping property in the borderline case

$$I_\Omega^\alpha : L^{p,\lambda} \hookrightarrow \mathrm{BMO}, \quad \alpha = \frac{n-\lambda}{p} \text{ (R. and S., 2019)}$$

Theorem (Pointwise estimate for the Riesz potential)

Let $1 < p < \infty$, $0 < \lambda < n$, $f \in L^{p-\varepsilon,\lambda}(\Omega)$, *and* $0 < \varepsilon < p-1$. *In the case* $\alpha = \frac{n-\lambda}{p}$ *the estimate*

$$|I_\Omega^\alpha f(x)| \leqslant C \left(r^\alpha Mf(x) + \frac{1}{\varepsilon} r^{-\frac{\alpha\varepsilon}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon,\lambda}(\Omega)} \right) \tag{8}$$

holds, where C *does not depend on* f *and* ε.

Sketch of the Proof

$$|I^\alpha_\Omega f(x)| \leqslant \int_{|x-y|<r} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,\mathrm{d}y + \int_{|x-y|>r} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,\mathrm{d}y = I_1(x) + I_2(x).$$

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$$\begin{aligned}
I_2(x) &\leqslant C \sum_{k=0}^{\infty} \|f\chi_{B(x,2^{k+1}r)}\|_{L^{p-\varepsilon}(\Omega)} \left(\int_{|y|>2^k r} \frac{\mathrm{d}y}{|y|^{(n-\alpha)(p-\varepsilon)'}} \right)^{\frac{1}{(p-\varepsilon)'}} \\
&\leqslant C \sum_{k=0}^{\infty} \|f\chi_{B(x,2^{k+1}r)}\|_{L^{p-\varepsilon}(\Omega)} \left(\int_{2^k r}^{\infty} \frac{\mathrm{d}\xi}{\xi^{1+\frac{(n-\lambda)\varepsilon+\lambda p}{p(p-\varepsilon-1)}}} \right)^{\frac{1}{(p-\varepsilon)'}} \\
&\leqslant C r^{-\frac{\alpha\varepsilon}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon,\lambda}(\Omega)} \sum_{k=0}^{\infty} 2^{-k\frac{\alpha\varepsilon}{p-\varepsilon}},
\end{aligned}$$

Theorem (Estimate for the Riesz potential of a function)

Let $1 \leqslant p < \infty$, $0 < \lambda < n$, *and* $f(x) := |x|^{\frac{\lambda - n}{p}} \in L^{p,\lambda}(B(0,R))$. *Then*

$$I_B^\alpha f(x) \simeq \ln \frac{1}{|x|} \quad \text{as} \quad x \to 0$$

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$L_{\psi}^{\infty),\lambda}$ spaces

Definition ($L_{\psi}^{\infty),\lambda}$ spaces)

$$\|f\|_{L_{\psi}^{\infty),\lambda}(\Omega)} := \sup_{q>1} \frac{1}{\psi(q)} \|f\|_{L^{q,\lambda}(\Omega)} < \infty. \tag{9}$$

$L_\psi^{\infty),\lambda}$ spaces

Theorem ($L_\psi^{\infty),\lambda}$ spaces)

Let Ω be a bounded open set, $0 < \lambda < n$, ψ satisfy the conditions (3) *and be doubling: $\psi(2t) \leqslant C\psi(t)$, $t > 0$. Then*

$$L_\psi^{\infty),\lambda}(\Omega) = L_\psi^{\infty)}(\Omega), \tag{10}$$

up to equivalence of norms.

Theorem (grand Morrey spaces)

Let Ω be a bounded set, $0 < \alpha < n$, $0 < \lambda < n$, and $p = \frac{n-\lambda}{\alpha}$. Then the operator I_Ω^α is bounded from the classical Morrey space $L^{p,\lambda}(\Omega)$ to $L_\theta^{\infty)}(\Omega)$ with $\theta = 1$ and this choice of θ is sharp.

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Jump:

Though $L^{p,\lambda}|_{\lambda=0} = L^{p}$, the above sharpness yields that the results for $L^{p,\lambda}$ does not turn into the result for L^{p} by taking $\lambda = 0$.

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where $\frac{1}{q_\varepsilon} = \frac{1}{p-\varepsilon} - \frac{\alpha}{n-\lambda} = \frac{1}{p-\varepsilon} - \frac{1}{p}$.

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$$\left(\frac{1}{t^\lambda}\int_{\Omega(x,t)} |I^\alpha_\Omega f(y)|^{q_\varepsilon}\,\mathrm{d}y\right)^{\frac{1}{q_\varepsilon}} \leqslant \frac{C}{\varepsilon}\|Mf\|^{\frac{\varepsilon}{p}}_{L^{p-\varepsilon,\lambda}(\Omega)}\|f\|^{\frac{p-\varepsilon}{p}}_{L^{p-\varepsilon,\lambda}(\Omega)}. \tag{12}$$

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$$\sup_{q>p'+\eta} \frac{1}{q^{1+\theta}}\|I^\alpha_\Omega\|_{L^{q,\lambda}(\Omega)} \leqslant C \sup_{0<\varepsilon<p-1-\delta} \varepsilon^\theta \|f\|_{L^{p-\varepsilon,\lambda}(\Omega)}, \tag{13}$$

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Under the assumptions of Theorem above, the operator I_Ω^α maps the Morrey space $L^{p,\lambda}(\Omega)$ into $\mathrm{BMO}(\Omega) \cap L_\theta^{\infty)}(\Omega)$, $\theta = 1$.

Definition (Morrey type spaces)

The Morrey type space $L^{p,q,\lambda}(\Omega)$ is defined as the set of measurable functions on Ω such that

$$\|f\|_{L^{p,q,\lambda}(\Omega)} = \sup_{x\in\Omega} \left(\int_0^\infty \left(\frac{1}{r^\lambda} \int_{\Omega(x,r)} |f(y)|^p \, \mathrm{d}y \right)^{\frac{q}{p}} \frac{\mathrm{d}r}{r} \right)^{\frac{1}{q}} < \infty, \quad 0 \in \Omega. \tag{14}$$

Definition ($L_{\psi}^{\infty),q,\lambda}$ spaces)

Let ψ satisfy the conditions (3). By $L_{\psi}^{\infty),q,\lambda}(\Omega)$ we denote the space of measurable functions such that

$$\|f\|_{L_{\psi}^{\infty),q,\lambda}(\Omega)} = \sup_{p>1} \frac{1}{\psi(p)} \|f\|_{L^{p,q,\lambda}(\Omega)} < \infty. \tag{15}$$

Theorem (Pointwise estimate for the Riesz potential)

Let $1 < p < \infty$, $1 \leqslant q < \infty$, $0 < \lambda < n$, $f \in L^{p-\varepsilon,q,\lambda}(\Omega)$, *and* $0 < \varepsilon < p-1$. *In the case* $\alpha = \frac{n-\lambda}{p}$, *the estimate*

$$|I_\Omega^\alpha f(x)| \leqslant C \left(r^\alpha Mf(x) + \frac{1}{\varepsilon^{\frac{1}{q'}}} r^{-\frac{\alpha\varepsilon}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon,q,\lambda}(\Omega)} \right) \tag{16}$$

holds, where C *does not depend on* f *and* ε.

Theorem (Riesz potential)

Let $0 < \lambda < n$, $1 \leqslant p < \infty$, $1 \leqslant q < \infty$, *and*

$$f_\beta(x) = \frac{1}{|x|^{\frac{n-\lambda}{p}} \left(\ln \frac{1}{|x|}\right)^{\beta}}.$$

Theorem (Riesz potential)

Let $0 < \lambda < n$, $1 \leqslant p < \infty$, $1 \leqslant q < \infty$, *and*

$$f_\beta(x) = \frac{1}{|x|^{\frac{n-\lambda}{p}} \left(\ln \frac{1}{|x|}\right)^\beta}.$$

If $\beta > \frac{1}{q}$, *then* $f_\beta \in L^{p,q,\lambda}(B(0, 1/2))$.

$L_{\psi}^{\infty),q,\lambda}$ spaces

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Let $0 < \lambda < n$, $1 \leqslant p < \infty$, $1 \leqslant q < \infty$, *and*

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If $\beta > \frac{1}{q}$, *then* $f_\beta \in L^{p,q,\lambda}(B(0, 1/2))$.
If $\frac{1}{q} < \beta < 1$, *then*

$$I_B^\alpha f_\beta \notin \bigcup_{0<\mu<n} L_\theta^{\infty),q,\mu}(B(0, 1/2))$$

when $\theta < 1 - \beta + \frac{1}{q}$.

Theorem (grand Morrey type spaces)

Let $1 < p < \infty$, $1 \leqslant q < \infty$, $0 < \mu < \lambda < n$, $\theta \geqslant 0$, *and* $\alpha = \frac{n-\lambda}{p}$. *Then the operator* I_{Ω}^{α} *is bounded from the classical Morrey type space* $L^{p,q,\lambda}(\Omega)$ *to* $L_{\xi}^{\infty),q,\mu}(\Omega)$, *where* $\xi = 1$.

$L_{\psi}^{\infty),q,\lambda}$ spaces

Theorem (grand Morrey type spaces)

Let $1 < p < \infty$, $1 \leqslant q < \infty$, $0 < \mu < \lambda < n$, $\theta \geqslant 0$, *and* $\alpha = \frac{n-\lambda}{p}$. *Then the operator* I_Ω^α *is bounded from the classical Morrey type space* $L^{p,q,\lambda}(\Omega)$ *to* $L_\xi^{\infty),q,\mu}(\Omega)$, *where* $\xi = 1$.
The operator I_Ω^α *is also bounded from the grand space* $L_\theta^{p),q,\lambda}(\Omega)$ *to the space* $L_\xi^{\infty),q,\mu}(\Omega)$, *where* $\xi = 1 + \theta$. *The choice of* ξ *in both cases is sharp.*

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