

Quantitative results on continuity of the spectral factorization mapping

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Notation

\mathbb{T} – unit circle, \mathbb{D} – unit disk:

$$\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}, \quad \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}.$$

Hardy spaces:

$$H^p(\mathbb{T}) := \{g \in L^p(\mathbb{T}) \mid g_n = 0 \text{ for } n < 0\}, \quad 1 \leq p \leq \infty,$$

where g_n is the n -th Fourier coefficient of g ;

$$H^p(\mathbb{D}) := \{g \text{ analytic in } \mathbb{D} \mid \|g\|_{H^p(\mathbb{D})} := \sup_{0 < r < 1} \|g(r \cdot)\|_{L^p(\mathbb{T})} < \infty\},$$
$$1 \leq p \leq \infty.$$

$H^p(\mathbb{T}) \rightleftharpoons$ boundary values of functions in $H^p(\mathbb{D})$.

Take any $g \in H^p(\mathbb{D}) \setminus \{0\}$. Let $a_n \in \mathbb{D}$ be the zeros of g . Consider the Blaschke product

$$B(z) = z^m \prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}, \quad z \in \mathbb{D}.$$

B has the same zeros as $g \implies g_0 := g/B$ is analytic in \mathbb{D} and has no zeros $\implies \log g_0$ is analytic in \mathbb{D} .

$|B| = 1$ a.e. on $\mathbb{T} \implies |g_0| = |g|$ a.e. on \mathbb{T}

Schwarz integral formula for $\log f_0 \implies$

$$\log g_0(z) \stackrel{\text{not quite!}}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log |g(e^{it})| dt, \quad z \in \mathbb{D}$$

and hence

$$\begin{aligned} g(z) &= B(z)g_0(z) = B(z) \exp(\log g_0(z)) \\ &\stackrel{\text{not quite!}}{=} B(z) \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log |g(e^{it})| dt\right), \quad z \in \mathbb{D}. \end{aligned}$$

Canonical factorisation in Hardy spaces

$$g(z) = \underbrace{B(z) S(z)}_{\text{inner function}} G(z),$$

where

$$B(z) = z^m \prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z} \quad (\text{Blaschke product}),$$

$$G(z) = \exp \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log |g(e^{it})| dt \right) \quad (\text{outer function}),$$

$$S(z) = \exp \left(- \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \right) \quad (\text{singular inner function}),$$

μ is a bounded nondecreasing singular function: $\mu'(t) = 0$ a.e.

$$|S| = 1 = |B| \text{ a.e. on } \mathbb{T} \implies |G| = |g| \text{ a.e. on } \mathbb{T}.$$

$$|g(z)| \leq |G(z)| \text{ for all } z \in \mathbb{D}.$$

$g \in H^p(\mathbb{D})$ is outer if and only if $g\mathcal{P}^+(\mathbb{C})$ is dense in $H^p(\mathbb{D})$,
 $1 \leq p < \infty$.

Here $\mathcal{P}^+(\mathbb{C})$ is the set of analytic polynomials

$$\mathcal{P}^+(\mathbb{C}) := \left\{ \sum_{k=0}^n c_k z^k : c_0, \dots, c_n \in \mathbb{C}, n \in \mathbb{N} \right\}.$$

Note that

$$g \in H^p(\mathbb{D}) \setminus \{0\} \implies \log |g| \in L^1(\mathbb{T}).$$

Suppose $f \in L^1(\mathbb{T})$, $f > 0$ a.e., and

$$\log f \in L^1(\mathbb{T}).$$

Then there exists a unique **outer** function $f^+ \in H^2(\mathbb{T})$ such that $f^+(0) > 0$ and

$$f(\zeta) = |f^+(\zeta)|^2, \quad \zeta \in \mathbb{T}.$$

$$f^+(z) = \exp \left(\frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log f(e^{i\theta}) d\theta \right), \quad |z| < 1.$$

Let $\mathcal{S}_N(\mathbb{T})$ denote the class of matrix functions F such that $F \in (L^1(\mathbb{T}))^{N \times N}$ is a.e. positive definite and

$$\log \det F \in L^1(\mathbb{T}).$$

If $F \in \mathcal{S}_N(\mathbb{T})$, then it admits a unique **spectral factorisation**:

$$F(\zeta) = F^+(\zeta) (F^+(\zeta))^*, \quad \zeta \in \mathbb{T},$$

where $F^+ \in (H^2(\mathbb{T}))^{N \times N}$ is an **outer** matrix function and $F^+(0)$ is positive definite (N. Wiener, 1955; H. Helson and D. Lowdenslager, 1958).

A matrix function $A \in (H^2(\mathbb{T}))^{N \times N}$ is called **outer** if $\det A \in H^{2/N}(\mathbb{T})$ is an outer function.

Equivalently, $A \in (H^2(\mathbb{T}))^{N \times N}$ is outer iff the set of products $A\mathcal{P}^+(\mathbb{C}^N)$ is dense in $(H^2(\mathbb{T}))^N$.

Here $\mathcal{P}^+(\mathbb{C}^N)$ is the set of \mathbb{C}^N -valued analytic polynomials

$$c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0, \quad c_0, \dots, c_n \in \mathbb{C}^N, \quad n \in \mathbb{N}.$$

Continuity of the spectral factorisation mapping

It is well known that

$$\|F_n - F\|_{L^1} \rightarrow 0 \not\Rightarrow \|F_n^+ - F^+\|_{H^2} \rightarrow 0,$$

but

$$\|F_n - F\|_{L^1} \rightarrow 0 \quad \text{and}$$

$$\int_{-\pi}^{\pi} \log \det F_n(e^{i\theta}) d\theta \rightarrow \int_{-\pi}^{\pi} \log \det F(e^{i\theta}) d\theta$$

\Downarrow

$$\|F_n^+ - F^+\|_{H^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

(G. Janashia, E. Lagvilava, and L. Ephremidze, 1999, 2011;
S. Barclay, 2004)

Quantitative results?

Question:

Does there exist

$$\Pi : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$$

such that $\Pi(\alpha, \beta) \rightarrow 0$ as $\alpha, \beta \rightarrow 0$, and

$$\|F_n^+ - F^+\|_{H^2} \leq \Pi(\|F_n - F\|_{L^1}, \|\log \det F_n - \log \det F\|_{L^1})?$$

Scalar case

Suppose $f \in L^1(\mathbb{T})$, $f > 0$ a.e., and

$$\log f \in L^1(\mathbb{T}).$$

Then there exists a unique **outer** function $f^+ \in H^2(\mathbb{T})$ such that $f^+(0) > 0$ and

$$f(\zeta) = |f^+(\zeta)|^2, \quad \zeta \in \mathbb{T}.$$

$$f^+(z) = \exp \left(\frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log f(e^{i\theta}) d\theta \right), \quad |z| < 1,$$

$$f^+(e^{i\vartheta}) = \exp \left(\frac{1}{2} \left(\log f(e^{i\vartheta}) + i(\log f)^\sim(e^{i\vartheta}) \right) \right), \quad \vartheta \in (-\pi, \pi],$$

where " \sim " denotes the harmonic conjugate:

$$\tilde{h}(e^{i\vartheta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(e^{i\theta}) \cot \left(\frac{\vartheta - \theta}{2} \right) d\theta.$$

Negative results

Theorem.

There exist functions $f_n, g_n \geq 0$, $n \in \mathbb{N}$ such that

$$\|f_n\|_{L_1}, \|g_n\|_{L_1} \leq 1, \quad \|f_n - g_n\|_{L_1} \leq \frac{1}{n}, \quad \|\log f_n - \log g_n\|_{L_1} \leq \frac{1}{n},$$

but $\|f_n^+ - g_n^+\|_{H^2} \geq 2 - 1/n$.

Remark. The norms $\|\log f_n\|_{L_1}$ and $\|\log g_n\|_{L_1}$ might not be bounded in the above theorem. One can change the estimates

$$\|f_n\|_{L_1}, \|g_n\|_{L_1} \leq 1 \text{ in the theorem for } \|f_n\|_{L_1} = 2\pi + 1, \\ \|g_n\|_{L_1} \leq 2\pi + 1, \|\log f_n\|_{L_1} \leq 1.$$

Positive results?

Question:

Can one get a positive result if one puts additional restrictions on f ?

Every function $f \in L^1(\mathbb{T})$ belongs to an Orlicz space ...

A crash course on Orlicz spaces

Let (Ω, Σ, μ) be a measure space and let $\Psi : [0, +\infty) \rightarrow [0, +\infty)$ be a non-decreasing function.

We want to define a Banach space of functions $f : \Omega \rightarrow \mathbb{C}$ (or \mathbb{R}) such that

$$\int_{\Omega} \Psi(|f(x)|) d\mu(x) < \infty. \quad (*)$$

If $\Psi(t) \equiv t^p$, then this is just $L_p(\Omega)$.

Difficulties:

- If Ψ is rapidly increasing, e.g. exponentially increasing, then the set of functions satisfying $(*)$ is not a linear space: $(*)$ does not imply that the same integral for $2f$ is finite.
- How do we define a norm? (What is a Ψ -analogue of taking the p -th power root of the integral?)

A crash course on Orlicz spaces

Definition

A continuous **convex** function $\Psi : [0, +\infty) \rightarrow [0, +\infty)$ is called an **N -function** if

$$\lim_{t \rightarrow 0^+} \frac{\Psi(t)}{t} = 0 \quad \text{and} \quad \lim_{t \rightarrow +\infty} \frac{\Psi(t)}{t} = +\infty.$$

The function $\Phi : [0, +\infty) \rightarrow [0, +\infty)$ defined by

$$\Phi(t) := \sup_{s \geq 0} (st - \Psi(s))$$

is called **complementary** to Ψ .

Examples of complementary N -functions:

$$\Psi(t) = \frac{t^p}{p}, \quad 1 < p < \infty, \quad \Phi(t) = \frac{t^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

$$\mathcal{A}(t) = e^t - 1 - t, \quad \mathcal{B}(t) = (1 + t) \ln(1 + t) - t.$$

A crash course on Orlicz spaces

Let Φ and Ψ be mutually complementary N -functions.

Norms:

$$\|f\|_{\Psi} = \|f\|_{\Psi, \Omega} := \sup \left\{ \left| \int_{\Omega} fg \, d\mu \right| : \int_{\Omega} \Phi(|g|) \, d\mu \leq 1 \right\},$$

$$\|f\|_{(\Psi)} = \|f\|_{(\Psi, \Omega)} := \inf \left\{ \kappa > 0 : \int_{\Omega} \Psi \left(\frac{|f|}{\kappa} \right) \, d\mu \leq 1 \right\}.$$

These norms are equivalent:

$$\|f\|_{(\Psi)} \leq \|f\|_{\Psi} \leq 2\|f\|_{(\Psi)}.$$

The Orlicz space:

$$L_{\Psi}(\Omega) := \{f \mid \|f\|_{\Psi, \Omega} < \infty\}.$$

Let K be the best constant in Kolmogorov's weak type $(1, 1)$ estimate

$$m\{\vartheta \in [-\pi, \pi) : |\tilde{\psi}(\vartheta)| \geq \lambda\} \leq \frac{K}{\lambda} \int_{-\pi}^{\pi} |\psi(\vartheta)| d\vartheta,$$
$$\lambda > 0, \quad \psi \in L_1[-\pi, \pi),$$

where m stands for the Lebesgue measure on the real line

(B. Davis, 1974:

$$K = \frac{1 + 3^{-2} + 5^{-2} + \dots}{1 - 3^{-2} + 5^{-2} - \dots} \approx 1.347),$$

and let

$$K_0 := \frac{K}{2} \int_0^{\pi} \frac{\sin \lambda}{\lambda} d\lambda.$$

Let Φ be an N -function and let

$$\Lambda_{\Phi}(s) := \inf \left\{ t > 0 : \frac{1}{t} \Phi' \left(\frac{1}{t} \right) \leq \frac{1}{s} \right\}, \quad s > 0.$$

Then

$$\Lambda_{\Phi}(s) \rightarrow 0 \text{ as } s \rightarrow 0 +.$$

and

$$\Phi(\tau) \equiv \frac{\tau^q}{q}, \quad 1 < q < \infty \implies \Lambda_{\Phi}(s) \equiv s^{1/q}.$$

Positive results

Theorem.

For every pair Φ and Ψ of mutually complementary N -functions, the following estimate holds

$$\|f^+ - g^+\|_{H^2}^2 \leq 2\|f - g\|_{L_1} + 4\|f\|_{\Psi} \Lambda_{\Phi} \left(\frac{K_0}{2} \|\log f - \log g\|_{L_1} \right).$$

Corollary.

For every $p \in (1, \infty)$, there exists a constant $C(p)$ such that

$$\|f^+ - g^+\|_{H^2}^2 \leq 2\|f - g\|_{L_1} + C(p)\|f\|_{L_p} \|\log f - \log g\|_{L_1}^{\frac{p-1}{p}}.$$

One can take $C(p) = 2^{\frac{p+1}{p}} K_0^{\frac{p-1}{p}} \left(\frac{p}{p-1} \right)^{\frac{p-1}{p}}$.

Corollary.

There exists a constant C such that

$$\|f^+ - g^+\|_{H^2}^2 \leq 2\|f - g\|_{L_1} + C\|f\|_{L_\infty} \|\log f - \log g\|_{L_1}.$$

One can take $C = 2K_0 < 2.5$.

The power $\frac{p-1}{p}$ in the estimate

$$\|f^+ - g^+\|_{H^2}^2 \leq 2\|f - g\|_{L_1} + C(p)\|f\|_{L_p}\|\log f - \log g\|_{L_1}^{\frac{p-1}{p}}$$

is optimal.

Theorem.

Let $p > 1$. For every $\gamma > \frac{p-1}{p}$, there exist functions $f, f_k \in \mathcal{S}_1(\mathbb{T}) \cap L_p(\mathbb{T})$, $k = 1, 2, \dots$, such that

$$\|f - f_k\|_{L_1} \rightarrow 0 \quad \text{and} \quad \|\log f - \log f_k\|_{L_1} \rightarrow 0,$$

while

$$\frac{\|f^+ - f_k^+\|_{H^2}^2}{\|\log f - \log f_k\|_{L_1}^\gamma} \rightarrow \infty.$$

Back to matrix functions

The negative result we had in the scalar case is of course correct in the matrix case too.

For positive results, we will need the following notation. For $F \in \mathcal{S}_n(\mathbb{T})$, let

$$\ell_F := \log \det F - n \log_+ \|F\| \in L_1(\mathbb{T}),$$

where $\log_+ x = \max(0, \log x)$ (note that $\ell_F \leq 0$ a.e.), and

$$Q_F := \frac{(\max\{1, \|F\|\})^n}{\det F} = e^{-\ell_F} = e^{|\ell_F|}.$$

Below, we will use conditions like $\ell_F \in L_{p_1}(\mathbb{T})$ and $Q_F \in L_{p_1}(\mathbb{T})$.

In the scalar case $n = 1$, one has $\ell_F = \log_+ \frac{1}{F}$ and $Q_F = \max \left\{ 1, \frac{1}{F} \right\}$. So, these conditions measure not how large F is but how strong its zeros are.

In the matrix case, these conditions control comparative sizes of the eigenvalues of F . More precisely, they control how small the determinant of F is compared to the maximum between 1 and the n -th power of the norm of F .

Reminder: $\|f^+ - g^+\|_{H^2}^2 \leq 2\|f - g\|_{L_1} + C(p)\|f\|_{L_p}\|\log f - \log g\|_{L_1}^{\frac{p-1}{p}}$

Theorem.

Suppose $F \in \mathcal{S}_n(\mathbb{T}) \cap L_{p_0}(\mathbb{T})^{n \times n}$, $G \in \mathcal{S}_n(\mathbb{T})$, and $\ell_F \in L_{p_1}(\mathbb{T})$, $p_0, p_1 \in (1, \infty)$, $\|G - F\|_{L_1} \leq 1$. Then for any $\alpha \in (0, 1)$, the following estimate holds

$$\begin{aligned} & \|G^+ - F^+\|_{H^2}^2 \leq 4\|G - F\|_{L_1} + 2q_0^{\frac{1}{q_0}} \left(\|F\|_{L_{p_0}} + 1 \right) \\ & \times \left[c(p_0) \left\| \log_+ \|G\| - \log_+ \|F\| \right\|_{L_1}^{\frac{1}{q_0}} + 2(2p_0)^{\frac{1}{p_0}} \left(3n\|G - F\|_{L_1}^{1-\alpha} \right. \right. \\ & \quad \left. \left. + 2(n+1)\alpha^{1-p_1} p_1 \|\ell_F\|_{L_{p_1}}^{p_1} \left| \log \|G - F\|_{L_1} \right|^{1-p_1} \right. \right. \\ & \quad \left. \left. + \left\| \log \frac{\det G}{\det F} \right\|_{L_1} \right)^{\frac{1}{q_0}} \right], \end{aligned}$$

where

$$c(p_0) := 2^{\frac{p_0+1}{p_0}} K_0^{\frac{1}{q_0}}, \quad q_0 := \frac{p_0}{p_0 - 1}.$$

Reminder: $\|f^+ - g^+\|_{H^2}^2 \leq 2\|f - g\|_{L_1} + C\|f\|_{L_\infty} \|\log f - \log g\|_{L_1}$

Theorem (continued)

Furthermore, if in addition $F \in L_\infty(\mathbb{T})^{n \times n}$, then the above inequality can be modified as

$$\begin{aligned} & \|G^+ - F^+\|_{H^2}^2 \leq 4\|G - F\|_{L_1} + 4 \max\{\|F\|_{L_\infty}, 1\} \\ & \times \left(K_0 \left\| \log_+ \|G\| - \log_+ \|F\| \right\|_{L_1} + 3n \|G - F\|_{L_1}^{1-\alpha} \right. \\ & \left. + 2(n+1)\alpha^{1-p_1} p_1 \|\ell_F\|_{L_{p_1}}^{p_1} \left| \log \|G - F\|_{L_1} \right|^{1-p_1} + \left\| \log \frac{\det G}{\det F} \right\|_{L_1} \right). \end{aligned}$$

The exponent $1 - p_1$ in the above theorem cannot be improved beyond $-p_1$ even for 2×2 matrix functions.

Theorem.

Suppose $F \in S_n(\mathbb{T}) \cap L_{p_0}(\mathbb{T})^{n \times n}$, $G \in S_n(\mathbb{T})$, and $Q_F \in L_{p_1}(\mathbb{T})$, $p_0, p_1 \in (1, \infty)$. If $\|F - G\|_{L_1} \leq e^{-4}$, then

$$\begin{aligned} & \|G^+ - F^+\|_{H_2}^2 \leq 4\|G - F\|_{L_1} \\ & + 2q_0^{\frac{1}{q_0}} \left(\|F\|_{L_{p_0}} + 1 \right) \left[c(p_0) \left\| \log_+ \|G\| - \log_+ \|F\| \right\|_{L_1}^{\frac{1}{q_0}} \right. \\ & + 2(2p_0)^{\frac{1}{p_0}} \left(\left(3n + \frac{4(n+1)\|Q_F\|_{L_{p_1}}^{2p_1}}{p_1+1} \left| \log \|G - F\|_{L_1} \right| \right) \|G - F\|_{L_1}^{\frac{p_1}{p_1+1}} \right. \\ & \left. \left. + \left\| \log \frac{\det G}{\det F} \right\|_{L_1} \right)^{\frac{1}{q_0}} \right], \end{aligned}$$

where $c(p_0)$ and q_0 are as in the previous theorem.

Theorem (continued)

Furthermore, if in addition $F \in L_\infty(\mathbb{T})^{n \times n}$, then the above inequality can be modified as

$$\begin{aligned} & \|G^+ - F^+\|_{H_2}^2 \leq 4\|G - F\|_{L_1} \\ & + 4 \max\{\|F\|_{L_\infty}, 1\} \left[K_0 \left\| \log_+ \|G\| - \log_+ \|F\| \right\|_{L_1} \right. \\ & + \left(3n + \frac{4(n+1)\|Q_F\|_{L^{p_1}}^{2p_1}}{p_1 + 1} \left| \log \|G - F\|_{L_1} \right| \right) \|G - F\|_{L_1}^{\frac{p_1}{p_1+1}} \\ & \left. + \left\| \log \frac{\det G}{\det F} \right\|_{L_1} \right]. \end{aligned}$$

The exponent $\frac{p_1}{p_1+1} = 1 - \frac{1}{p_1+1}$ in the above theorem cannot be improved beyond $\frac{2p_1}{2p_1+1} = 1 - \frac{1}{2p_1+1}$ even for 2×2 matrix functions.

Let

$$R_{\Psi}(\tau) := \tau \Psi^{-1} \left(\frac{4}{\tau} \right), \quad \tau > 0.$$

Then

$$R_{\Psi}(\tau) \rightarrow 0 \text{ as } \tau \rightarrow 0+$$

and

$$\frac{1}{2} \Lambda_{\Phi}(\tau) < R_{\Psi}(\tau) \leq 8 \Lambda_{\Phi}(\tau), \quad \forall \tau > 0.$$

Theorem.

Let $F, G \in \mathcal{S}_n(\mathbb{T})$, and let (Φ_0, Ψ_0) and (Φ_1, Ψ_1) be two pairs of mutually complementary N -functions such that

$$F \in L_{\Psi_0} \text{ and } \ell_F \in L_{\Psi_1}.$$

Then for any nondecreasing function $\nu : [0, \infty) \rightarrow [0, 1]$ satisfying

$$\nu(\tau) \rightarrow 0 \text{ and } \tau/\nu(\tau) \rightarrow 0 \text{ as } \tau \rightarrow 0+$$

the following estimate holds

$$\begin{aligned} & \|G^+ - F^+\|_{H_2}^2 \leq 4\|G - F\|_{L_1} \\ & + 2 \left(\|F\|_{\Psi_0} + \Phi_0^{-1}(1) \right) \left[4\Lambda_{\Phi_0} \left(\frac{K_0}{2} \left\| \log_+ \|G\| - \log_+ \|F\| \right\|_{L_1} \right) \right. \\ & \left. + R_{\Psi_0} \left(\frac{6n\|G - F\|_{L_1}}{\nu(\|G - F\|_{L_1})} + \frac{4(n+1)|\log \nu(\|G - F\|_{L_1})|}{\Psi_1\left(\frac{|\log \nu(\|G - F\|_{L_1})|}{\|\ell_F\|_{(\Psi_1)}}\right)} + 2 \left\| \log \frac{\det G}{\det F} \right\|_{L_1} \right) \right]. \end{aligned}$$

Theorem.

Let $F, G \in \mathcal{S}_n(\mathbb{T})$, and let (Φ_0, Ψ_0) be a pair of mutually complementary N -functions. Suppose $F \in L_{\Psi_0}$ and $\ell_F \in L_\infty$. Then

$$\begin{aligned} & \|G^+ - F^+\|_{H_2}^2 \leq 4\|G - F\|_{L_1} \\ & + 2 \left(\|F\|_{\Psi_0} + \Phi_0^{-1}(1) \right) \left[4\Lambda_{\Phi_0} \left(\frac{K_0}{2} \left\| \log_+ \|G\| - \log_+ \|F\| \right\|_{L_1} \right) \right. \\ & \left. + R_{\Psi_0} \left(\left(2e^{\|\ell_F\|_{L_\infty}} + 1 \right) n \|G - F\|_{L_1} + \left\| \log \frac{\det G}{\det F} \right\|_{L_1} \right) \right]. \end{aligned}$$

Theorem.

Let $F \in \mathcal{S}_n(\mathbb{T}) \cap L_\infty(\mathbb{T})^{n \times n}$ and $\ell_F \in L_\infty(\mathbb{T})$. Then

$$\|G^+ - F^+\|_{H_2}^2 \leq \|F\|_{L_\infty} \left(n e^{\|\ell_F\|_{L_\infty}} \|G - F\|_{L_1} + \left\| \log \frac{\det G}{\det F} \right\|_{L_1} \right).$$