On the regularity of characteristic functions in Nikol'skii-Besov Spaces

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1. Introduction

Let $E \subset \mathbb{R}^d$ be a measurable set with $0 < |E| < \infty$. Then

$$\mathcal{X}_E \in L_p(\mathbb{R}^d) \qquad \text{and} \qquad \mathcal{X}_E \notin W_p^1(\mathbb{R}^d)\,.$$

In addition

$$\mathcal{X}_E \notin C^s(\mathbb{R}^d) \quad \text{for any } s > 0$$

Hence, we need different spaces of fractional order of smoothness. Here we will deal with Nikol'skii-Besov spaces, but we could have chosen also Lizorkin-Triebel spaces.

2. Nikol'skii-Besov spaces

$0 < p, q \le \infty$, $s \in \mathbb{R}$: $\quad B^s_{p,q}(\mathbb{R}^d) \ldots$ (Fourier analysis, differences, atoms, wavelets, φ-transform)

$0 < \theta < 1$, $1 \le p, q \le \infty$, $m \in \mathbb{N}$:

$$(W^{m-1}_p(\mathbb{R}^d), W^m_p(\mathbb{R}^d))_{\theta,q} = B^s_{p,q}(\mathbb{R}^d)\,, \quad s := m - 1 + \theta\,.$$

$\mathrm{Tr}: \; f(x) \mapsto f(0, x_2, \ldots, x_d)$

$$1 < p < \infty: \qquad \mathrm{Tr}: \; W^m_p(\mathbb{R}^d) \longrightarrow B^{m-1/p}_{p,p}(\mathbb{R}^{d-1})$$

Differences

$$\Delta_h^1 f(x) := f(x+h) - f(x)\,, \qquad \Delta_h^{m+1} f(x) := \Big(\Delta_h^1(\Delta_h^m f)\Big)(x)$$

Let $m \in \mathbb{N}$, $0 < s < m$. Then $f \in L_p(\mathbb{R}^d)$ belongs to $B^s_{p,q}(\mathbb{R}^d)$ if and only if

$$\bigg(\int_{\substack{h \in \mathbb{R}^d \\ 0<|h|<1}} \Big[|h|^{-s} \, \| \Delta_h^m f \,|L_p(\mathbb{R}^d)\| \Big]^q \frac{dh}{|h|^d} \bigg)^{1/q} < \infty \,.$$

$$\begin{aligned} \| f \,|B^s_{p,q}(\mathbb{R}^d)\| & := & \| f \,|L_p(\mathbb{R}^d)\| \\ & & + \bigg(\int_{\substack{h \in \mathbb{R}^d \\ 0<|h|<1}} \Big(|h|^{-s} \, \| \Delta_h^m f \,|L_p(\mathbb{R}^d)\| \Big)^q \frac{dh}{|h|^d} \bigg)^{1/q} \end{aligned}$$

A first example

<u>$1 \le p \le \infty$, $0 < s < 1$:</u>

$f \in L_p(\mathbb{R}^d)$ belongs to $B^s_{p,\infty}(\mathbb{R}^d)$ if

$$\sup_{0<|h|<1} |h|^{-s}\Big(\int_{\mathbb{R}^d} |f(x+h) - f(x)|^p\, dx\Big)^{1/p} < \infty\,.$$

$\mathcal{X}$ - characteristic function of $(0,1)$, $1 \le p < \infty$, $0 < h < 1$:

$$\int_{-\infty}^{\infty} |\mathcal{X}(x+h) - \mathcal{X}(x)|^p\, dx = \int_{-h}^{0} 1^p\, dx + \int_{1-h}^{1} 1^p\, dx = 2h\,.$$

<u>$1 < p < \infty$:</u> $\qquad \mathcal{X} \in B^s_{p,\infty}(\mathbb{R}) \qquad \Longleftrightarrow \qquad 0 < s \le 1/p\,.$

3. How smooth are characteristic functions ?

3.1. Balls and cuboids

We put $F := \mathbb{R}^d \setminus E$. For $h \in \mathbb{R}^d$ we define

$$\begin{aligned} E(h) &:= \{x \in E : \; x+h \notin E\}\,; \\ F(h) &:= \{x \in F : \; x+h \notin F\}\,. \end{aligned}$$

$$\begin{aligned} \|\, \mathcal{X}_E(\,\cdot + h) - \mathcal{X}_E(\,\cdot\,)\,|L_p(\mathbb{R}^d)\|^p &= \int_{E(h)} 1\, dx + \int_{F(h)} 1\, dx \\ &= |E(h)| + |F(h)|\,. \end{aligned}$$

Lemma

Let $1 \leq p < \infty$ *and* $0 < s < 1$. *Then* $\mathcal{X}_E$ *belongs to* $B^s_{p,\infty}(\mathbb{R}^d)$ *if and only if*

$$\sup_{|h|<1} |h|^{-s} \left(|E(h)| + |F(h)|\right)^{1/p} < \infty \,. \tag{1}$$

There is an easy but interesting consequence of Lemma 1.

Lemma

Let $1 < p < \infty$ *and* $0 < s < 1$. *Then* $\mathcal{X}_E \in B^s_{1,\infty}(\mathbb{R}^d)$ *implies* $\mathcal{X}_E \in B^{s/p}_{p,\infty}(\mathbb{R}^d)$ *and vice versa.*

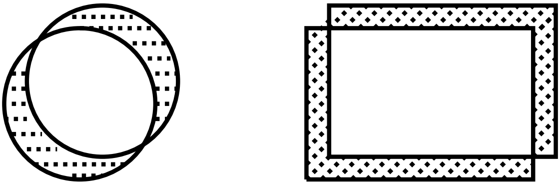


Fig. 1

Let E be a cube or a ball in $\mathbb{R}^d$. Then it is easily seen that $|E(h)| + |F(h)| \asymp |h|$, $|h| < 1$.

Lemma

Let $1 \le p < \infty$ and $s > 0$.
(i) *The characteristic function $\mathcal{X}$ belongs to $B^s_{p,\infty}(\mathbb{R})$ if and only if $s \le 1/p$.*
(ii) *Let $d \ge 2$. Then the characteristic function $\mathcal{X}_E$ of either a ball or a cuboid, i.e., the cartesian product of d segments, belongs to $B^s_{p,\infty}(\mathbb{R}^d)$ if and only if $s \le 1/p$.*

3.2. The limiting case $s = 1/p$

Titchmarsh ($d = 1$), Gulisashvili ($d > 1$):

Lemma

Let B be a ball in $\mathbb{R}^d$. A locally integrable function u satisfying

$$\lim_{h \to 0} \frac{1}{|h|} \int_B |u(x+h) - u(x)|\, dx = 0$$

is constant a.e. on B.

$$\int_B |\mathcal{X}_E(x+h) - \mathcal{X}_E(x)|\, dx = \int_B |\mathcal{X}_E(x+h) - \mathcal{X}_E(x)|^p\, dx$$

Theorem

Let $1 \le p < \infty$ and $1 \le q < \infty$. Then there exists no subset $E \subset \mathbb{R}^d$, $0 < |E| < \infty$, such that $\mathcal{X}_E \in B^{1/p}_{p,q}(\mathbb{R}^d)$.

Let $1 \le p < \infty$, $1 \le q < \infty$ and $s > 1/p$. Then

$$B^s_{p,\infty}(\mathbb{R}^d) \hookrightarrow B^{1/p}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{1/p}_{p,\infty}(\mathbb{R}^d)\,.$$

Within the class of Nikol'skii-Besov spaces the classes $B^{1/p}_{p,\infty}(\mathbb{R}^d)$ are the smallest in which we can find nontrivial characteristic functions, or, with other words, $B^{1/p}_{p,\infty}(\mathbb{R}^d)$ represents the maximal regularity, a characteristic function may have.

Recall $BV(\mathbb{R}) \cap L_1(\mathbb{R}) \hookrightarrow B^1_{1,\infty}(\mathbb{R})$. Clearly $\mathcal{X} \in BV(\mathbb{R})$.

Extension to $d \ge 1$: A locally integrable function $f:\ \mathbb{R}^d \to \mathbb{R}$ is of bounded variation if its first order partial derivatives (in the distributional sense) are bounded Borel measures. The space $BV \cap L_1(\mathbb{R}^d)$ will be endowed with the norm

$$\| \, f \, |BV \cap L_1(\mathbb{R}^d)\| := \sum_{j=1}^d \left| \frac{\partial f}{\partial x_j} \right| + \| \, f \, |L_1(\mathbb{R}^d)\|\,.$$

where $|\frac{\partial f}{\partial x_j}|$ denotes the total variation of the measure.

$\mathcal{H}^s$-s-dimensional Hausdorff measure.

Definition

Then the perimeter of a set E is the quantity

$$\mathrm{per}\, E := \liminf_{j\to\infty} \mathcal{H}^{d-1}(\partial M_j)\,,$$

where the limit is taken with respect to all sequences $(M_j)_j$ of sets with a smooth boundary (or polyhedra) such that

$$\lim_{j\to\infty} \|\, \mathcal{X}_E - \mathcal{X}_{M_j}\, |L_1(\mathbb{R}^d)\| = 0\,.$$

Kronrod-Federer-Fleming-Rishel formula:

$$\|\, f\, |BV(\mathbb{R}^d)\| = \int_{-\infty}^{\infty} \mathrm{per}\,(\{x \in \mathbb{R}^d :\ f(x) > t\})\, dt\,,$$

Fleming, Rishel 1960, Burago, Zalgaller 1988.

$$\mathcal{X}_E \in BV(\mathbb{R}^d) \qquad \text{if and only if} \qquad \mathrm{per}\, E < \infty\,. \tag{2}$$

Hardy, Littlewood ($d = 1$), Gulisashvili ($d \ge 1$) 1984:

Lemma

Let E be a measurable set in $\mathbb{R}^d$ with $|E| < \infty$. Then there exist positive constants c_1, c_2 such that

$$c_1 \sup_{t>0} t^{-1} \omega_1(t, \mathcal{X}_E) \le per\, E \le c_2 \sup_{t>0} t^{-1} \omega_1(t, \mathcal{X}_E)$$

where

$$\omega_1(t, \mathcal{X}_E) := \sup_{|h|<t} \|\Delta_h^1 \mathcal{X}_E \,|L_1(\mathbb{R}^d)\| , \qquad t > 0 .$$

The main result in the limiting case $s = 1/p$.

Theorem

Let $E \subset \mathbb{R}^d$ and $0 < |E| < \infty$. Then the following assertions are equivalent:

(i) $\operatorname{per} E < \infty$;

(ii) $\sup_{|h|<1} |h|^{-1} \left(|E(h)| + |F(h)|\right) < \infty$;

(iii) $\mathcal{X}_E \in BV(\mathbb{R}^d)$;

(iv) $\mathcal{X}_E \in B^{1/p_0}_{p_0,\infty}(\mathbb{R}^d)$ *for some* p_0, $1 \le p_0 < \infty$.

(v) $\mathcal{X}_E \in B^{1/p}_{p,\infty}(\mathbb{R}^d)$ *for all* p, $1 \le p < \infty$.

- Gulisashvili 1984/1985
- Maz'ya/Shaposnikova 1985.

3.3. Examples

$x = (x', x_d),\ x' = (x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1},\ x_d \in \mathbb{R}.$

Definition

Let $d \geq 2$. An open bounded set E is called elementary Lipschitz domain if there exist a function φ and numbers $0 < D_1 \leq D_2 < \infty$, $a_1, \ldots a_d$, $b_1, \ldots, b_{d-1}, L$ such that

(i) $\operatorname{diam}(E) \leq D_2$;

(ii) $E = \{x \in \mathbb{R}^d :\ a_d < x_d < \varphi(x'),\ x' \in W\}$;

(iii) $W := \{x' \in \mathbb{R}^{d-1} :\ a_i < x_i < b_i,\ i = 1, \ldots, d-1\}$;

(iv) $a_d + D_1 \leq \varphi(x'),\ x' \in W$;

(v) $|\varphi(x') - \varphi(y')| \leq L\,|x' - y'|, \quad x', y' \in W$.

Lemma

Let E be an elementary Lipschitz domain. Then $\mathcal{X}_E \in BV \cap B^{1/p}_{p,\infty}(\mathbb{R}^d)$ for all $p \in [1, \infty)$.

Basic properties of Nikol'skii-Besov spaces yield the following.

Corollary

Let E be a domain which can be written as the union of the closures of a finite number of pairwise disjoint domains $E_1, \ldots, E_N$ such that any of the E_j, $j = 1, \ldots, N$, is the image of an elementary Lipschitz domain under a finite number of rotations, translations and reflections. Then $\mathcal{X}_E \in BV \cap B^{1/p}_{p,\infty}(\mathbb{R}^d)$ for all $p \in [1, \infty)$.

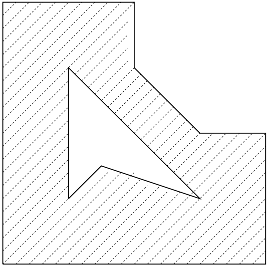


Fig. 2

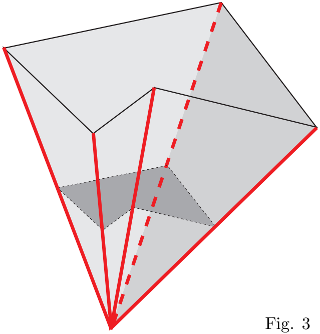


Fig. 3

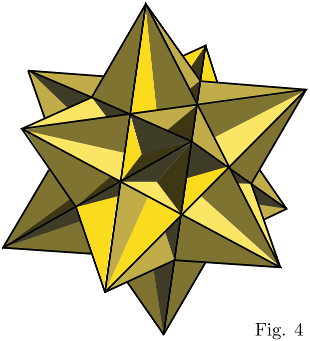


Fig. 4

A special polyhedral domain which is not Lipschitz

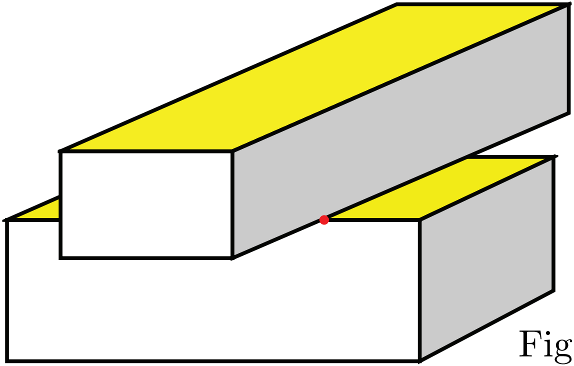


Fig. 8

The Astroid and the rotated Astroid

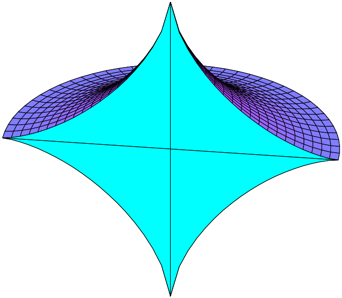


Fig. 5

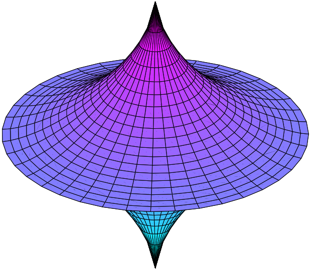


Fig. 6

Lipschitz boundary versus maximal regularity

- Let $0 < \varepsilon < 1$. The domains

$$\Omega_\varepsilon := \{(x,y) \in \mathbb{R}^2 : \ |x| < 1 \, , \quad |x|^\varepsilon < |y| < 1\}$$

have finite perimeter.

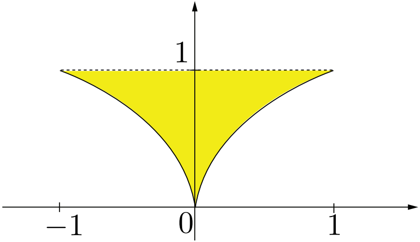


Fig. 7

3.4. The case $s < 1/p$ - necessary conditions

Lemma

Let E be a bounded domain. If $\mathcal{X}_E \in B^s_{p,q}(\mathbb{R}^d)$ for some $s > 0$, $1 \le p < \infty$ and $1 \le q \le \infty$, then $|\partial E| = 0$ follows.

• There exists a further necessary condition in terms of the packing dimension by Jaffard, Meyer 1996.

3.5. The case $s < 1/p$ - sufficient conditions

For positive δ we define

$$\partial E^\delta := \{x \in \mathbb{R}^d : \ \mathrm{dist}(x, \partial E) \le \delta\} \, .$$

Definition

The s-dimensional upper Minkowski content of a set $A \subset \mathbb{R}^d$ is defined by

$$\mathcal{M}^{*s}(A) := \limsup_{\delta \downarrow 0} \, (2\delta)^{s-d} \, |A^\delta|$$

$$\begin{aligned} E(h) &:= \{x \in E : \ x + h \notin E\} \, ; \\ F(h) &:= \{x \in F : \ x + h \notin F\} \, . \end{aligned}$$

$$\Big(E(h) \cup F(h)\Big) \subset \partial E^\delta \, , \quad |h| = \delta \, .$$

Lemma

Let $E \subset \mathbb{R}^d$ *such that* $0 < |E| < \infty$*. Let* $1 \le p < \infty$, $0 < s \le 1$ *and* $0 < a \le 1$*.*

(i) *If*

$$\sup_{0<\delta<a} \delta^{-s} \, |\partial E^\delta| < \infty \, ,$$

then $\mathcal{X}_E \in B^{s/p}_{p,\infty}(\mathbb{R}^d)$*.*

(ii) *If the* $d - s$*-dimensional upper Minkowski content of* ∂E*, i.e.,* $\mathcal{M}^{*(d-s)}(\partial E)$*, is finite, then* $\mathcal{X}_E \in B^{s/p}_{p,\infty}(\mathbb{R}^d)$*.*

- Strichartz (Madych) 1993;
- Jaffard and Meyer 1996;
- Runst and Sickel 1996;
- Sickel 1999;

Approximation by piecewise constant functions

For $j \in \mathbb{N}_0$ and $k \in \mathbb{Z}^d$ we put

$$Q_{j,k} := \left\{ x \in \mathbb{R}^d : \; 2^{-j} k_i \le x_i < 2^{-j}(k_i + 1), \; i = 1, \ldots, d \right\}.$$

For $f \in L_p(\mathbb{R}^d)$ we define

$$E_j(f)_p \; := \; \inf \Big\{ \| f - g \,| L_p(\mathbb{R}^d)\| : \; g \in L_p(\mathbb{R}^d) \; \text{ and } \; g \; \text{ is constant on the dyadic cubes } \; Q_{j,k}, \; k \in \mathbb{Z}^d \Big\}, \; j \in \mathbb{N}_0 .$$

Let $1 \le p < \infty$, $1 \le q \le \infty$ and $0 < s < 1/p$. Then $f \in B^s_{p,q}(\mathbb{R}^d)$ if and only if $f \in L_p(\mathbb{R}^d)$ and

$$\Big(\sum_{j=0}^{\infty} \big[2^{js} \, E_j(f)_p \big]^q \Big)^{1/q} < \infty \,, \tag{3}$$

. . ., Kashin, Saakyan (1989), Oswald (1995), . . .

For a subset E of $\mathbb{R}^d$ and $\delta > 0$ we put

$$\partial E_+^\delta := \{x \in E : \operatorname{dist}(x, \partial E) \le \delta\}, \tag{4}$$

Theorem

Let E be a bounded domain in $\mathbb{R}^d$. Let $1 \le p < \infty$, $1 \le q \le \infty$ and $0 < s < 1/p$. Suppose either

$$\int_0^1 \delta^{-sq} \, |\partial E_+^\delta|^{q/p} \frac{d\delta}{\delta} < \infty \qquad \textit{if} \quad q < \infty$$

or

$$\sup_{0<\delta<1} \delta^{-s} \, |\partial E_+^\delta|^{1/p} < \infty \qquad \textit{if} \quad q = \infty .$$

Then $\mathcal{X}_E \in B^s_{p,q}(\mathbb{R}^d)$ holds.

Quasiballs

A homeomorphism $f: \mathbb{R}^d \to \mathbb{R}^d$ is called *K-quasiconformal* if there is a constant $K < \infty$ such that for all $x \in \mathbb{R}^d$

$$K(x) := \limsup_{\varepsilon \to 0} \frac{\max_{a:\ |x-a|=\varepsilon} |f(x) - f(a)|}{\min_{b:\ |x-b|=\varepsilon} |f(x) - f(b)|} \le K \, .$$

A K-quasiball is the image of the unit ball under a K-quasiconformal mapping. For $d = 2$ also the name quasicircle is commonly used.

Theorem

Let $1 \le p < \infty$, $0 < s < 1$ and let $E \subset \mathbb{R}^d$ be a K-quasiball. Then

$$\| \mathcal{X}_E \, | B^s_{p,p}(\mathbb{R}^d) \| \asymp \Big(|E| + \int_0^{\delta^*} \delta^{-ps} \, |\partial E^\delta| \, \frac{d\delta}{\delta} \Big)^{1/p} ,$$

where $\delta^* := \inf\{\delta : \ E \subset \partial E^\delta\}$.

- Faraco and Rogers 2012.

3.6. Examples

Definition

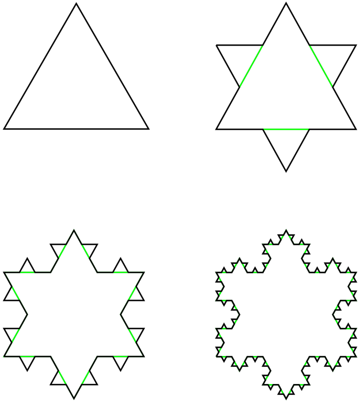
Let $d \geq 2$. We define an elementary domain with Hölder continuous boundary of order $\alpha \in (0,1]$ by replacing (v) in Definition 2 by

$$|\varphi(x') - \varphi(y')| \leq L\,|x' - y'|^{\alpha}\,, \quad x', y' \in W\,.$$

Lemma

Let $d \geq 2$. Let $\alpha \in (0,1)$. Let E be an elementary domain with Hölder continuous boundary of order α. Then $\mathcal{X}_E \in B^{\alpha/p}_{p,\infty}(\mathbb{R}^d)$ for all $p \in [1,\infty)$.

The snow flake domain



Ω - snow flake domain with the von Koch curve as its boundary.

(i) Ω is a (ε, ∞) domain;

(ii) Ω is a John domain;

(iii) Ω is a quasiball;

(iv) Ω is a selfsimilar set;

(v) $\dim_H \partial\Omega = \dim_M \partial\Omega = \log 4/\log 3$;

- Jones 1981;
- Falconer 1990;
- Mattila 1995;
- Buckley, Koskela 1995;
- D. Meyer 2010;

Corollary

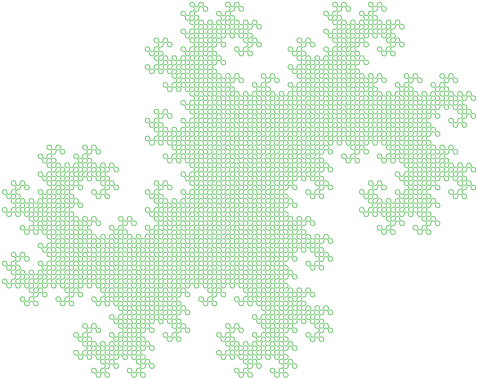
Let $1 \leq p < \infty$. The characteristic function $\mathcal{X}_\Omega$ of the snowflake domain belongs to $B^s_{p,p}(\mathbb{R}^2)$ if and only if $s < \frac{1}{p}\left(2 - \frac{\log 4}{\log 3}\right)$.

- S. 1999 (if-part);
- Jaffard, Meyer 1996;
- Faraco, Rogers 2013 (if and only if).

Theorem

Let $1 \leq p < \infty$. The characteristic function $\mathcal{X}_\Omega$ of the snowflake domain Ω belongs to $B^s_{p,\infty}(\mathbb{R}^2)$ if and only if $s \leq \big(2 - \log 4/\log 3\big)/p$.

The twindragon



- a space filling curve with a fractal boundary.
- Relatives: heighway dragon, Levy dragon.
- can be used as a scaling function in wavelet theory

Let T be the set filled by this curve and denote by ∂T its boundary. It holds that $\dim_H \partial T = \dim_M \partial T = \delta$, where δ is the unique solution of

$$\Big(\frac{1}{\sqrt{2}}\Big)^{\delta} + 2\Big(\frac{1}{2\sqrt{2}}\Big)^{\delta} = 1\,,$$

given by

$$\delta := \log_2\left(\frac{1+\sqrt[3]{73-6\sqrt{87}}+\sqrt[3]{73+6\sqrt{87}}}{3}\right) \sim 1.5236\,,$$

- Mandelbrot 1987.

Lemma

Let $1 \le p < \infty$. Then we have $\mathcal{X}_T \in B^{\frac{2-\delta}{p}}_{p,\infty}(\mathbb{R}^2)$.

d-sets (Ahlfors regular sets)

Definition

Let Γ be a non-empty compact set in $\mathbb{R}^d$. Let $0 \le \delta \le d$. Then Γ is called an δ-set if there exists a finite Radon measure μ in $\mathbb{R}^d$ satisfying

$$\operatorname{supp}\mu = \Gamma \qquad \text{and} \qquad \mu(B(y,r)) \asymp r^{\delta}, \quad y \in \Gamma, \quad 0 < r < 1.$$

Corollary

Let E be a bounded domain in $\mathbb{R}^d$ with boundary ∂E being an δ-set for some $d-1 < \delta < d$. Let $1 \le p < \infty$. Then we have $\mathcal{X}_E \in B^{\frac{d-\delta}{p}}_{p,\infty}(\mathbb{R}^d)$ for all $p \in [1,\infty)$.

- Triebel 2003;
- Schneider, Vybiral 2013.

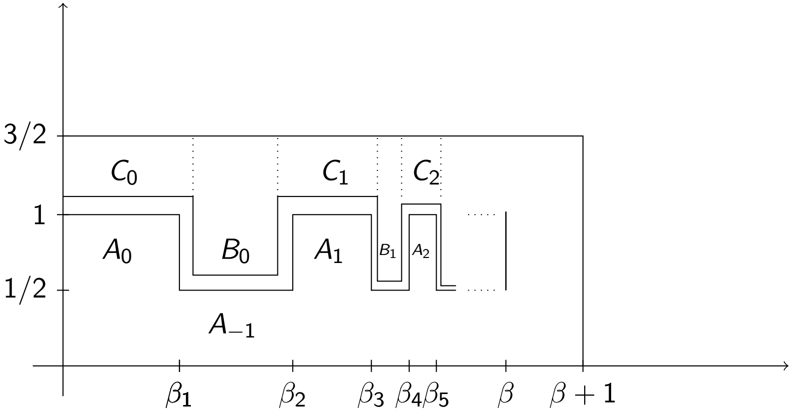
Particular examples of δ-sets are self-similar sets. E.g., ∂T and $\partial \Omega$ are self-similar sets. In view of these examples one may conjecture: $\mathcal{X}_E \in B^s_{p,\infty}(\mathbb{R}^d)$ and

$$s = \frac{1}{p}\Big(d - \dim_M \partial E\Big) = \frac{1}{p}\Big(d - \dim_H \partial E\Big) .$$

But this is wrong in general !

Let $\gamma \geq \alpha > 1$. Then we define

$$\beta_j := \sum_{\ell=1}^{j} \ell^{-\alpha}\,, \qquad \beta := \sum_{\ell=1}^{\infty} \ell^{-\alpha} \quad \text{and} \quad \delta_j := \frac{1}{4(2j+2)^\gamma}\,, \qquad j \in \mathbb{N}\,.$$



Properties of the sets $E_{\alpha,\gamma}$

Theorem

Let $\gamma > \alpha > 1$. Then the sets $E_{\alpha,\gamma}$ have the following properties.

(i) $\dim_M(\partial E_{\alpha,\gamma}) = 1 + 1/\alpha$.

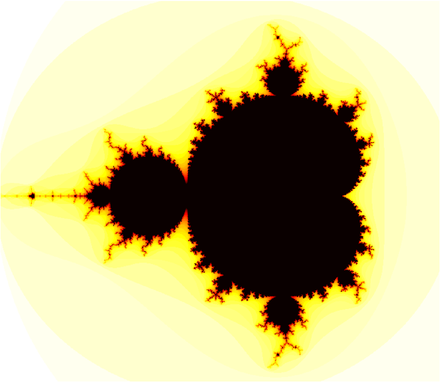
(ii) $\dim_H(\partial E_{\alpha,\gamma}) = \dim_P(\partial E_{\alpha,\gamma}) = 1$.

(iii) $\chi_{E_{\alpha,\gamma}} \in B^s_{p,\infty}(\mathbb{R}^2)$ *if and only if* $s\,p \le (1 - 1/\gamma)$.

(iv) *Let* $1 \le q < \infty$. *Then* $\chi_{E_{\alpha,\gamma}} \in B^s_{p,q}(\mathbb{R}^2)$ *if and only if* $s\,p < (1 - 1/\gamma)$.

$$\begin{aligned}\frac{1}{p}\Big(d - \dim_M \partial E_{\alpha,\gamma}\Big) &= \frac{1}{p}\Big(2 - (1 + \frac{1}{\alpha})\Big) < \frac{1}{p}\Big(1 - \frac{1}{\gamma}\Big) \\ &< \frac{1}{p}\Big(2 - 1\Big) = \frac{1}{p}\Big(d - \dim_H \partial E_{\alpha,\gamma}\Big).\end{aligned}$$

4. The Mandelbrot set



The Mandelbrot set D has the following properties:

- $\dim_H D = 2$;
- $\dim_H \partial D = 2$;
- $\dim_M \partial D = 2$, see Shishikura 1998.

5. The necessary condition of Jaffard and Meyer

We define

$$\partial E_* = \Big\{ x \in \partial E \, : \, \exists \mu > 0 \quad \text{such that} \ \forall \varepsilon, \, 0 < \varepsilon \le 1, \, \exists A_\varepsilon \, , B_\varepsilon \, \text{satisfying}$$

$$A_\varepsilon \subset B(x,\varepsilon) \cap E, \ B_\varepsilon \subset B(x,\varepsilon) \cap F, \quad \text{and} \quad |A_\varepsilon| \cdot |B_\varepsilon| \ge \mu \, \varepsilon^{2d} \Big\} .$$

• If E is an (ε, δ)-domain, then $\partial E = \partial E_*$.

Let A be a subset of $\mathbb{R}^d$. By $\dim_P(A)$ we denote the packing dimension of A.

Jaffard, Meyer (1996):

Theorem

Let E be a nontrivial subset of $\mathbb{R}^d$. Suppose $\mathcal{X}_E$ belongs to $B^s_{p,p}(\mathbb{R}^d)$ for some $s > 0$ and $1 \le p < \infty$. Then $\dim_P(\partial E_*) \le d - sp$.

6. Some references

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