Oversampling in symmetric, regular de Branges spaces

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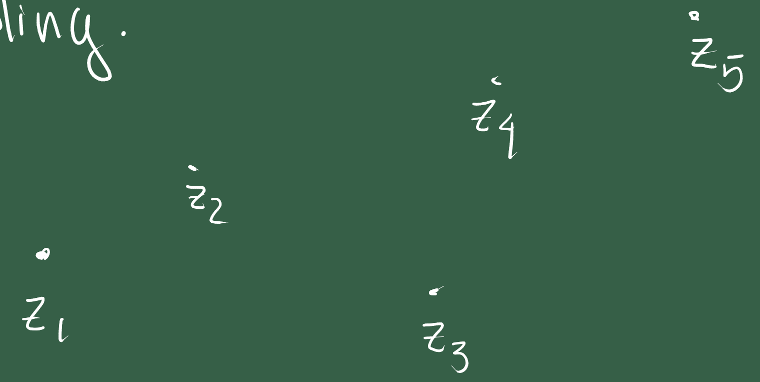
Sampling.

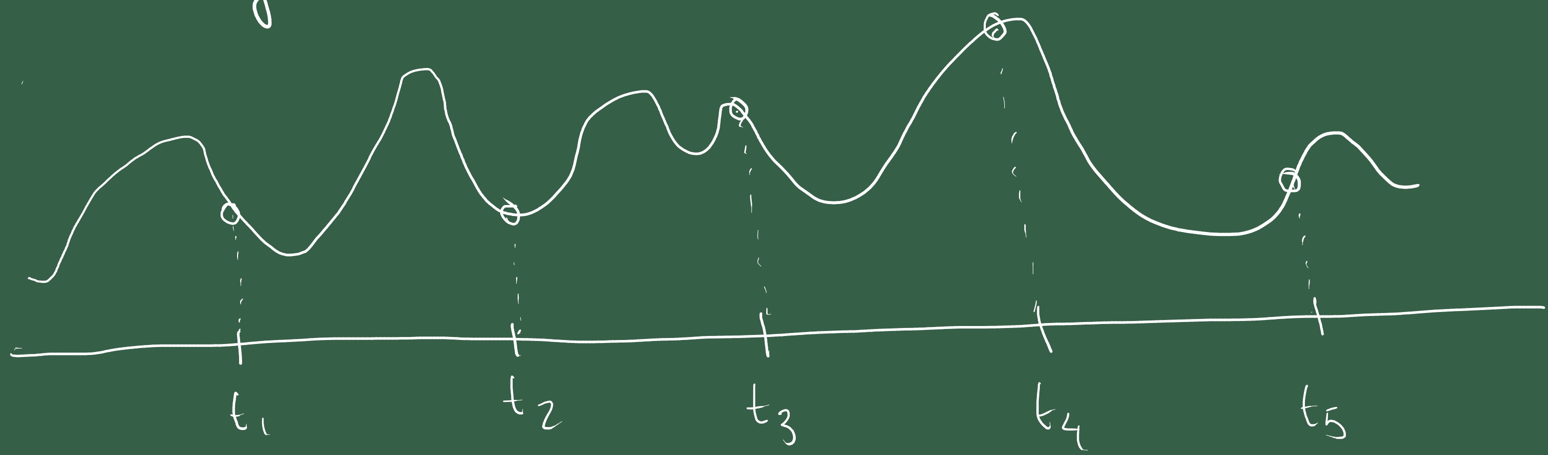


$+$

Space of functions

$$f(z_k) \xrightarrow{\text{reconstruct}} f.$$

Signal analysis:



The PW space:

$$PW_a := \left\{ \int_{-a}^{a} f(x)\, e^{-izx}\, dx \; : \; f \in L_2(-a,a) \right\}$$

Sampling theory was actually developed in this space

If $f \in PW_a \Rightarrow$

$$f(z) = \sum G_a\left(z, \frac{n\pi}{a}\right) f\left(\frac{n\pi}{a}\right) \qquad G_a(z,t) = \frac{\sin(a(z-t))}{a(z-t)}$$

dB spaces

RKHS – Hilbert space of functions s.t. the evaluation functional

$$F \mapsto F(z) \quad \forall z$$

is continuous.

$\Rightarrow \exists \ K_z : \quad F(z) = \langle K_z, F \rangle$

K_z is the reproducing kernel of the space

Def. A reproducing kernel Hilbert space of entire functions is a de Branges space when it is isometrically invariant under the maps

a) $F(z) \longmapsto F^{\#}(z) \quad \left(F^{\#}(z) = \overline{F(\bar{z})} \right)$

b) $F(z) \longmapsto \dfrac{z - \bar{w}}{z - w} F(z)$ $w \in \mathbb{C}$ is a zero of F if any.

Alternatively, let E be an $\mathcal{H}-\mathcal{B}$ function

(i.e an entire function s.t. $|E(z)| > |E(\bar{z})| \ \forall \ z \in \mathbb{C}_+$)

Def. $\mathcal{B}(E) := \left\{ F \text{ entire} : \frac{F}{E}, \frac{F^{\#}}{E} \in H^2(\mathbb{C}_+) \right\}$

endowed with the inner product of $\mathcal{H}^2(\mathbb{C}_+)$.

For any $E \in \mathcal{HB}$ $\mathcal{B}(E)$ is a dB space and for any dB space $\mathcal{B}$, there is an $\mathcal{HB}$ function E s.t. $\mathcal{B} = \mathcal{B}(E)$.

Def. A dB space is regular when it is closed under the map $F(z) \longmapsto \frac{F(z) - F(w)}{z - w}$ $w \in \mathbb{C} \setminus \mathbb{R}$

$\boxed{T}$ If $\mathcal{B}(E)$ is regular, then E is zero-free on the real line.

Def. A dB space is symmetric when it is isometrically invariant under the map

$$F(z) \longmapsto F(-z) \qquad z \in \mathbb{C}.$$

$\boxed{T}$ If $\mathcal{B}(E)$ is symmetric, then

$$E^{\#}(z) = E(-z) \qquad z \in \mathbb{C}$$

Def. A dB subspace of $\mathcal{B}$ is a subspace $\mathcal{A} \subset \mathcal{B}$ that itself is a dB space (edowed with the inner product of $\mathcal{B}$)

$\boxed{T}$ All dB subspaces of a dB space are totally ordered by inclusion.

$\Leftrightarrow$ If $\mathcal{A}_1$ and $\mathcal{A}_2$ ard dB subspaces of $\mathcal{B}$

$\Rightarrow$ either $\mathcal{A}_1 \subset \mathcal{A}_2$ or $\mathcal{A}_2 \subset \mathcal{A}_1$

$$\mathrm{Sub}(\mathcal{B}) := \{ \mathcal{A} \subset \mathcal{B} : \mathcal{A} \text{ is a dB space} \}$$

$\uparrow$

This is the maximal chain of nested dB subspaces of $\mathcal{B}$.

$\boxed{T}$ All the spaces in $\mathrm{Sub}(\mathcal{B})$ are regular if and only if one space in $\mathrm{Sub}(\mathcal{B})$ is.

An important property of regular dB spaces is that their functions are of exponential type.

$$\tau(F) := \limsup_{R \to \infty} \frac{\max_{|z|=R} \log |F(z)|}{R}$$

Exponential type $\iff$ $\tau(F) < +\infty$

The exponential type is uniformy bounded across $\mathcal{B}$.

$$\tau(\mathcal{B}) := \sup_{F \in \mathcal{B}} \tau(F) < +\infty$$

$\boxed{\mathrm{I}}$ If $\mathcal{B} = \mathcal{B}(E) \implies \tau(\mathcal{B}) = \tau(E)$

$\mathcal{B}$ is of normal type when $\tau(\mathcal{B}) > 0$

and of minimal type when $\tau(\mathcal{B}) = 0$

The type of a dB space will play an important role in our considerations.

The operator of multiplication by the independent variable

Def. $\operatorname{dom} S_{\mathcal{B}} := \{ F \in \mathcal{B} : zF(z) \in \mathcal{B} \}$

$$(S_{\mathcal{B}} F)(z) := zF(z) \qquad \forall\ F \in \operatorname{dom} S_{\mathcal{B}}.$$

This operator has many interesting properties.

$\boxed{T}$ The operator $S_{\mathcal{B}}$ has a one parameter family of canonical self-adjoint extensions $S_{\mathcal{B}}^{\gamma}$ $\gamma \in [0, \pi)$, $\sigma(S_{\mathcal{B}}^{\gamma})$ is discrete and simple.

Disclaimer: In general $S_{\mathcal{B}}$ is not densely defined, but for this talk, we assume that $S_{\mathcal{B}}$ is densely defined.

Since:

$$\langle k_w, (S_{\mathcal{B}} - wI)F \rangle = 0$$

$$\Rightarrow \quad k_w \in \ker(S_{\mathcal{B}}^* - \bar{w} I) \quad \forall\, w \in \mathbb{C}.$$

$$\Rightarrow \quad \lambda \in \sigma(S_{\mathcal{B}}^r) \Rightarrow \quad k_\lambda \text{ is an eigenfunction of } S_{\mathcal{B}}^r$$

$$F = \sum_{\lambda \in \mathfrak{G}_\gamma} \left\langle \frac{K_\lambda}{\|K_\lambda\|}, F \right\rangle \frac{K_\lambda}{\|K_\lambda\|}$$

$$F(z) = \langle K_z, F \rangle = \left\langle K_z, \sum_{\lambda \in \mathfrak{G}_\gamma} \left\langle \frac{K_\lambda}{\|K_\lambda\|}, F \right\rangle \frac{K_\lambda}{\|K_\lambda\|} \right\rangle$$

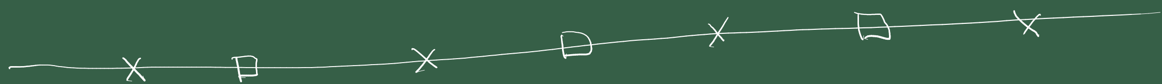
$$= \sum_{\lambda \in \mathfrak{G}_\gamma} \frac{\langle K_\lambda, F \rangle}{\| K_\lambda \|^2} \langle K_z, K_\lambda \rangle$$

$$= \sum_{\lambda \in \mathfrak{G}_\gamma} \frac{\langle K_z, K_\lambda \rangle}{\langle K_\lambda, K_\lambda \rangle} F(\lambda)$$

$$\mathcal{B} \xleftrightarrow{\text{bijection}} \ell_2(\mathcal{B}, \gamma)$$

$$\ell_2(\mathcal{B},\gamma) := \left\{ \{a_\lambda\}_{\lambda \in \sigma_\gamma} : \left\{ \frac{a_\lambda}{\langle K_\lambda, K_\lambda \rangle^{1/2}} \right\}_{\lambda \in \sigma_\gamma} \in \ell_2 \right\}$$

$$PW_a \longleftrightarrow \ell_2$$



Oversampling.

$\mathcal{B}$

$$l_\infty(\mathcal{B},\gamma) := \left\{ \{a_\lambda\}_{\lambda\in\sigma_\gamma} \ : \ \left\{ \frac{a_\lambda}{\langle K_\lambda, K_\lambda\rangle^{1/2}} \right\}_{\lambda\in\sigma_\gamma} \in l_\infty \right\}$$

$\mathcal{A} \in \mathrm{Sub}(\mathcal{B})$

$$F \in \mathcal{A} \Rightarrow \quad F(z) = \sum_{\lambda\in\sigma_\gamma} G_{\gamma,\mathcal{B}}(z,\lambda)\, F(\lambda)$$

$$F(z) = \sum_{\lambda\in\sigma_{\tilde\gamma}} G_{\tilde\gamma\mathcal{A}}(z,\lambda)\, F(\lambda)$$

$$F_\varepsilon(z) := \sum_{\lambda\in\sigma_\gamma} \widetilde{G}_{\gamma\mathcal{B}\mathcal{A}}(z,\lambda)\left(F(\lambda) + \varepsilon_\lambda\right)$$

where $\varepsilon := \{\varepsilon_\lambda\}_{\lambda\in\sigma_\gamma} \in l_\infty(\mathcal{B},\gamma)$

$$\Longrightarrow \quad |F(z) - F_\varepsilon(z)| \leq C(\mathbb{K})\, \|\varepsilon\|_{\ell_\infty(\mathcal{B},r)} \qquad z \in \mathbb{K} \qquad (*)$$

for any compact $\mathbb{K}$ of $\mathbb{C}$.

Def. A dB space $\mathcal{B}$ has the oversampling property relative to $\mathcal{A} \in \mathrm{sub}(\mathcal{B})$ when there is a sampling kernel $\widetilde{G}_{\mathcal{B},\mathcal{A},\gamma}$ s.t. $(*)$ holds.

It has been proven that the dB spaces:

PW

(2019) $\mathcal{B}$ generated by a regular Schrödinger operator

(2020) " " Bessel operator

have the oversampling property relative to any subspace in the corresponding chain.

Canonical systems and dB spaces

Given $\eta \in (0,\infty)$, consider

$$J\,y'(x) = -z\,H(x)\,y(x) \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$z \in \mathbb{C} \qquad x \in (0,\eta)$$

where $H : (0,\eta) \to \mathbb{R}^{2\times 2}$ is an integrable, essentially nonzero nonnegative ($\Rightarrow$ symmetric) matrix valued function.

Disclaimer: The setting here is completely general, but we do not touch upon singular intervals in this talk.

$L^2_H(0,1)$ is the linear set of equivalence classes of measurable vector-valued functions $f : (0,1) \to \mathbb{C}^2$

s.t.

$$\int_0^1 f(x)^* H(x)\, f(x)\, dx$$

$L^2_H(0,1)$ becomes a Hilbert space, once endowed with

$$\int_0^1 f^*(x)\, H(x)\, g(x)\, dx .$$

In this Hilbert space, consider the relation:

$$A_1 = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in L^2_H(0,1) \oplus L^2_H(0,1) \;:\; f \text{ has a representative } \mathring{f} \in AC[0,1] \right.$$
$$\left. \text{s.t. } J\mathring{f}'(x) = -H(x)\, g(x) \text{ a.e. } x \in (0,1) \text{ and } \mathring{f}_2(0) = 0 \right\}$$

$$A_{10} := \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in A_1 : \ \mathring{f}(1) = 0 \right\}$$

$$A_1^{\gamma} := \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in A_1 : \ \mathring{f}_1(1) \sin\gamma - \mathring{f}_2(1) \cos\gamma = 0 \right\}$$

$$A_{10} \subset A_1^{\gamma} \subset A_1$$

The solution $u(z,x)$ to the initial value problem $\odot$ with initial cond. $u(z,0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ satisfies

$$u(z,\cdot) \in \ker(A_1 - zI)$$

$$B_1 = \left\{ F(z) = \int_0^1 u(z,x)^T H(x) f(x)\, dx : \ f \in \operatorname{clos} \operatorname{dom} A_1 \right\}$$

$$\|F\|_{B_1} = \|f\|_{L^2_H(0,1)}$$

$\boxed{T}$ For any regular dB space there is a Hamiltonian $\mathcal{H}$ s.t. $\mathcal{B} = \mathcal{B}_l$ and $\mathcal{B}$ is symmetric $\iff$ $\mathcal{H}$ is a.e. diagonal.

$\boxed{T}$ Consider $\mathcal{B}_l$ with a diagonal Hamiltonian $\mathcal{H}$ and denote by $\tau(l) = \tau(\mathcal{B}_l)$. If τ' is absolutely continuos in a closed interval $I \subset (0,b]$, then there exists $a \in I$ s.t. $\mathcal{B}_b$ has the oversampling property relative to every $\mathcal{B}_l \in \mathrm{Sub}(\mathcal{B}_a)$.