On decay of spectral projections associated to Laplacians on certain Riemannian manifolds.

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Spectral projections for the Laplacian on $\mathbb{R}^n$

To begin with, let us consider the Fourier series of a 2π periodic function f on the real line

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx}$$

where $\hat{f}$ are the Fourier coefficients of the functions f.

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construct smooth functions whose Fourier coefficients have any given but arbitrary decay. However, this changes drastically once we impose some restrictions of the function f. For example, suppose f is known to vanish on an open interval. Then any exponential decay for the Fourier coefficients is ruled out. For, if $|\hat{f}(k)| \leq Ce^{-a|k|}$ for some $a > 0$ the series

$$f(x+iy) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ik(x+iy)}$$

converges and defines a holomorphic function in a neighbourhood of $\mathbb{R}$ leading to a contradiction.

So, it is natural to ask the following question: what is the best possible decay allowed for the Fourier coefficients of a nontrivial function which vanishes on a non-empty open set?

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In the context of Fourier transform on $\mathbb{R}$ and considering functions which are compactly supported this question has been addressed by Ingham (1934), Levinson (1936), Paley-Wiener (1934). They considered decay of the form $|\hat{f}(y)| \leq Ce^{-\psi(y)}$ under some conditions on ψ. It turns out that the condition

$$\int_1^\infty \psi(t)t^{-2}dt < \infty$$

is both necessary and sufficient for the existence of such functions.

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In 1934, Ingham treated the case where $\psi(y) = |y|\theta(y)$ where θ is an even function decreasing to 0 at infinity. He used Denjoy-Carleman theorem on quasi-analytic functions. Using Radon transform, Ingham's theorem for the Fourier transform on $\mathbb{R}^n$ was recently deduced by Bhowmik-Ray-Sen (2019).

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Ingham: Let $\theta(t)$ be a nonnegative even function on $\mathbb{R}$ such that $\theta(t)$ decreases to zero when $t \to \infty$. There exists a nonzero compactly supported continuous function f on $\mathbb{R}^n$, having Fourier transform $\widehat{f}$ satisfying the estimate $|\widehat{f}(\xi)| \leq Ce^{-|\xi|\theta(|\xi|)}$ if and only if θ satisfies $\int_1^\infty \theta(t)t^{-1}dt < \infty$.

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Another efficient way of proving Ingham's theorem is to use the following result.

Chernoff: Let f be a smooth function on $\mathbb{R}^n$. Let Δ be the standard Laplacian on $\mathbb{R}^n$. Assume that $(-\Delta)^m f \in L^2(\mathbb{R}^n)$ for all $m \in \mathbb{N}$ and

$$\sum_{m=1}^{\infty} \|(-\Delta)^m f\|_2^{-\frac{1}{2m}} = \infty.$$

If f and all its partial derivatives vanish at 0, then f is identically zero.

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If f and all its partial derivatives vanish at 0, then f is identically zero. To bring out the connection between Ingham and Chernoff theorems let us look at the inversion formula for the Fourier transform written in the form

$$f(x) = c_n \int_0^\infty f * \varphi_\lambda(x)\lambda^{n-1}d\lambda$$

where c_n is an explicit constant and $\varphi_\lambda(y)$ is the Bessel function

$$\varphi_\lambda(y) = J_{n/2-1}(\lambda|y|)(\lambda|y|)^{-n/2+1}.$$

Spectral projections for the Laplacian on $\mathbb{R}^n$

As $f * \varphi_\lambda$ are eigenfunctions of the Laplacian with eigenvalues $-\lambda^2$ the above also gives the explicit spectral decomposition of the Laplacian:

$$(-\Delta)f(x) = c_n \int_0^\infty \lambda^2 \, f * \varphi_\lambda(x) \lambda^{n-1} d\lambda$$

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The Plancherel theorem reads as

$$\int_{\mathbb{R}^n} |f(x)|^2 dx = d_n \int_0^\infty \Big(\int_{\mathbb{R}^n} f * \varphi_\lambda(x) \bar{f}(x) dx \Big) \lambda^{n-1} d\lambda$$

which is a consequence of the formula

$$\int_{\mathbb{R}^n} f * \varphi_\lambda(x) \bar{f}(x) dx = c_n' \int_{S^{n-1}} |\hat{f}(\lambda\omega)|^2 d\sigma(\omega).$$

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The above relation allows us to relate the decay of $\hat{f}(\lambda)$ and the decay of the spectral projections $f * \varphi_\lambda(x)$. To exploit this connection more effectively we make use of the spherical means of the function f.

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Recall that the spherical means of a function f on $\mathbb{R}^n$ are defned by

$$f * \sigma_r(x) = \int_{|y|=r} f(x-y)\, d\sigma_r(y)$$

where σ_r is the normalised surface measure on the sphere $|y| = r$ in $\mathbb{R}^n$.

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As the Fourier transform of σ_r is explicitly given by $\varphi_r(\xi)$ we can use Fourier inversion to write the above as

$$f * \sigma_r(x) = c_n \int_0^\infty f * \varphi_\lambda(x)\varphi_\lambda(r)\lambda^{n-1}d\lambda$$

We can consider the spherical means as a function on $\mathbb{R}^n \times \mathbb{R}^n$ which is radial in the second set of variables. Thus

$$u(x,y) = f * \sigma_{|y|}(x) = c_n \int_0^\infty f * \varphi_\lambda(x)\varphi_\lambda(y)\lambda^{n-1}d\lambda$$

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From the above it is clear that $(-\Delta)^m u(\cdot, y) = (-\Delta)^m u(x, \cdot)$ as both are equal to

$$c_n \int_0^\infty f * \varphi_\lambda(x)\varphi_\lambda(y)\lambda^{2m+n-1}d\lambda$$

Spectral projections for the Laplacian on $\mathbb{R}^n$

Yet another useful observation is that $\varphi_\lambda(y) = \varphi_r(\lambda)$, $r = |y|$ and hence $u(x, y)$ is the Hankel transform of the function $\lambda \to f * \varphi_\lambda(x)$.

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The Plancherel theorem for the Hankel transform allows us to conclude that

$$\|(-\Delta)^m u(x, \cdot)\|_2^2 = c_n \int_0^\infty |f * \varphi_\lambda(x)|^2 \lambda^{4m+n-1} d\lambda$$

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We can now draw some important conclusions from the above. Suppose the given function f vanishes on an open set V. Then for any $x \in V$ the spherical means $f * \sigma_r(x)$ vanishes for all small enough r, i.e. there exists $\delta(x) > 0$ such that $u(x, y) = 0$ for all $|y| < \delta(x)$.

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Suppose we further assume that the spectral projections $f * \varphi_\lambda(x)$ satisfy the following estimate:

$$\sup_{x \in V} |f * \varphi_\lambda(x)| \leq C e^{-\lambda\,\theta(\lambda)}.$$

If we assume that $\theta(\lambda) \geq c\lambda^{-1/2}$ for large λ and $\int_1^\infty \theta((t)t^{-1} dt = \infty$ it is possible to verify that

$$\sum_{m=0}^{\infty} \|(-\Delta)^m u(x, \cdot)\|_2^{-\frac{1}{2m}} = \infty.$$

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By appealing to Chernoff's theorem we can conclude that $u(x, y) = 0$ for all $y \in \mathbb{R}^n$ which means for all $x \in V$ and $r > 0$ we have

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But now the projections $f * \varphi_\lambda(x)$ are eigenfunctions of Δ and hence real analytic. This forces them to vanish for all $x \in \mathbb{R}^n$ and $\lambda > 0$. By inversion theorem for the Fourier transform we conclude finally that $f = 0$.

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We make two remarks concerning the above proof: (i) we have only assumed the decay assumption on $f * \varphi_\lambda(x)$ for $x \in V$, the set on which f vanishes and (ii) the Chernoff's theorem is needed only for radial functions.

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The idea of using spherical means allows us to treat similar results for spectral projections associated to several elliptic operators.

Laplacian on noncompact Riemannian symmetric spaces

Let us consider the case of the Laplace-Beltrami operator Δ on a rank one Riemannian symmetric space of noncompact type $X = G/K$. Here G is a semisimple Lie group and K a maximal compact subgroup. Thus functions on X are identified with right K-invariant functions on the group G.

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The spectral decomposition of the Laplace-Beltrami operator Δ on X is given by

$$f(g) = C \int_0^\infty f * \Phi_\lambda(g) |c(\lambda)|^{-2} d\lambda$$

where Φ_λ are the so-called spherical functions on X and $c(\lambda)$ is the celebrated c-function of Harish-Chandra.

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The spherical functions Φ_λ play the role of Bessel functions in the present context. They are (i) K-biinvariant and (ii) eigenfunctions of Δ with eigenvalues $-(\lambda^2 + \rho^2)$. Moreover, they are explicitly known as Jacobi functions.

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We use the decomposition $G = KAK, A = \mathbb{R}^+$ of the group to identify K-biinvariant functions with functions on $\mathbb{R}^+$.

Laplacian on noncompact Riemannian symmetric spaces

If f is a K-biinvariant function on G and if $g = ka_rk'$ then by abuse of notation we simply write $f(r) = f(g)$. The action of Δ on K-biinvariant functions reduces to the Jacobi operator: $\Delta f(g) = \mathcal{L}_{\alpha,\beta} f(r)$ where

$$\mathcal{L}_{\alpha,\beta} := \frac{d^2}{dr^2} + ((2\alpha+1)\coth r + (2\beta+1)\tanh r)\frac{d}{dr}.$$

Here α and β are parameters associated to the given symmetric space X.

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$$\Phi_\lambda(g) = \varphi_\lambda^{(\alpha,\beta)}(r), \;\; \Delta\Phi_\lambda(g) = -(\lambda^2+\rho^2)\Phi_\lambda(g).$$

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The role of the Hankel transform is played by the Jacobi transform defined for functions f on $\mathbb{R}^+$ by

$$\tilde{f}(\lambda) = \int_0^\infty f(r)\varphi_\lambda^{(\alpha,\beta)}(r)\tilde{w}_{\alpha,\beta}(r)\,dr$$

where the weight function is given by $\tilde{w}_{\alpha,\beta}(r) = (2\sinh r)^{2\alpha+1}(2\cosh r)^{2\beta+1}$.

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With an explicit function $c_{\alpha,\beta}$, the inversion formula for the Jacobi transform reads as

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Given a right K-invariant function f on G and $h \in G$ we define the spherical means

$$A_h f(g) = \int_K f(gkh)dk.$$

As f is right K-invariant, it follows that the function $u(g,h) = A_h f(g)$ is K-biinvariant whose spectral decomposition reads as (when $h = ka_r k'$)

$$u(g,h) = c\int_0^\infty f * \Phi_\lambda(g)\,\varphi_\lambda^{(\alpha,\beta)}(r)\,|c_{\alpha,\beta}(\lambda)|^{-2}d\lambda.$$

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Theorem: Let μ be a finite positive Borel measure on $\mathbb{R}$ for which all the moments $M(m) = \int_{-\infty}^{\infty} t^m d\mu, m \geq 0$ are finite. If we further assume that the moments satisfy the Carleman condition $\sum_{m=1}^{\infty} M(2m)^{-1/2m} = \infty$, then polynomials are dense in $L^p(\mathbb{R}, d\mu), 1 \leq p < \infty$.

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Here is a sketch of the proof of Chernoff theorem for the Jacobi operator, which is also valid for Bessel operators $B_\alpha = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r}$.

Laplacian on noncompact Riemannian symmetric spaces

Once we have a Chernoff theorem for the Jacobi operator, we can prove an Ingham theorem for the spectral projections. In order to prove such a theorem we make use of the following result of de Jeu.

Theorem: Let μ be a finite positive Borel measure on $\mathbb{R}$ for which all the moments $M(m) = \int_{-\infty}^{\infty} t^m d\mu, m \geq 0$ are finite. If we further assume that the moments satisfy the Carleman condition $\sum_{m=1}^{\infty} M(2m)^{-1/2m} = \infty$, then polynomials are dense in $L^p(\mathbb{R}, d\mu), 1 \leq p < \infty$.

Here is a sketch of the proof of Chernoff theorem for the Jacobi operator, which is also valid for Bessel operators $B_\alpha = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r}$.

For the sake of simplicity of notation let us write $L = \mathcal{L}_{\alpha,\beta}, c(\lambda) = c_{\alpha,\beta}(\lambda)$ and consider a function f on $\mathbb{R}^+$ satisfying the condition $\sum_{m=1}^{\infty} \|L^m f\|_2^{-\frac{1}{2m}} = \infty$.

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We consider the following measure μ_f defined on the Borel subsets of $\mathbb{R}$ by

$$\int_{-\infty}^{\infty} \varphi(t) d\mu_f(t) = \int_0^{\infty} \varphi(\lambda^2 + \rho^2) |\tilde{f}(\lambda)| |c(\lambda)|^{-2} d\lambda$$

for any Borel measurable function φ on $\mathbb{R}$. It then follows that

$$M(2m) = \int_{-\infty}^{\infty} t^{2m} d\mu_f(t) = \int_0^{\infty} (\lambda^2 + \rho^2)^{2m} |\tilde{f}(\lambda)| |c(\lambda)|^{-2} d\lambda.$$

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Using Plancherel theorem, the assumption of f translates into the Carlemen condition on the moments $M(m)$. Hence by de Jeu's theorem we conclude that polynomials are dense in $L^1(\mathbb{R}, d\mu_f)$.

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If we assume that f vanishes near 0 it follows from the inversion formula that

$$L^m f(r) = \frac{1}{2}\int_0^{\infty}(\lambda^2+\rho^2)^m\tilde{f}(\lambda)\varphi_\lambda(r)|c(\lambda)|^{-2}d\lambda = 0$$

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Letting $r \to 0$ and noting that $\varphi_\lambda(0) \neq 0$ we conclude that for any polynomial p

$$\int_0^{\infty} p(\lambda^2+\rho^2)\tilde{f}(\lambda)|c(\lambda)|^{-2} d\lambda = 0.$$

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The above along with the fact that polynomials are dense in $L^1(\mathbb{R}, d\mu_f)$ allows us to conclude that $f = 0$. Thus we have proved

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Theorem:Let θ be a positive decreasing function defined on $(0, \infty)$ that vanishes at infinity. Assume that $\int_1^\infty \theta(t)\, t^{-1} dt = \infty$. Let $f \in L^1(X)$ be a nontrivial function vanishing on an open set V. Then it is not possible to have the estimate

$$\sup_{x \in V} |f * \Phi_\lambda(x)| \leq Ce^{-\lambda\theta(\lambda)}.$$

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Both families B_α and $\mathcal{L}_{\alpha,\beta}$ have continuous spectrum. The same ideas can be used with operators having discrete spectrum. The typical examples we have in mind are related to expansions in terms of Jacobi and Laguerre polynomials.

Laplacian on compact Riemannian symmetric spaces

Let (G, K) be a compact symmetric pair and let $X = G/K$ be the associated symmetric space. We assume that X has rank one. Let $G = KAK$ be a Cartan decomposition of G where A is identified with $(0, R)$ for some $R > 0$. As before the elements of A will be denoted by $a_r, r > 0$.

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The spectral decomposition of the Laplace-Beltrami operator Δ on G/K is provided by the Peter-Weyl theorem:

$$f(g) = \sum_{\lambda \in \widehat{G}_K} d_\lambda \, f * \varphi_\lambda(g)$$

where $\widehat{G}_K$ is the subset of the unitary dual $\widehat{G}$ consisting of equivalence classes of class-1 representations of G and for each $\lambda \in \widehat{G}_K$, φ_λ is the associated zonal spherical function and d_λ is the dimension of the space on which λ is realised.

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The spherical functions $\varphi_\lambda(r)$ turn out to be Jacobi polynomials $P_\lambda^{(\alpha,\beta)}$ and they are eigenfunctions of Δ with eigenvalues c_λ. The normalised functions $d_\lambda^{1/2}\varphi_\lambda(r)$, $\lambda \in \widehat{G}_K$ form an orthonormal basis for $L^2(A, w(r)dr)$ for a suitable weight w.

Laplacian on compact Riemannian symmetric spaces

The spherical means of a function f on G is defined by

$$f(g,r) = \int_K \int_K f(gka_r k')dk\, dk'.$$

Observe that $f(g,r)$ is a right K-invariant function of $g \in G$ and hence we can consider it as a function on the symmetric space X.

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Moreover, if Δ_0 is the restriction of Δ to K-biinvariant functions, it follows that $\Delta_0 \varphi_\lambda(r) = c_\lambda \varphi_\lambda(r)$.

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As before a version of Chernoff's theorem for Δ_0 allows us to prove an Ingham theorem for the spectral projections associated to Δ.

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Instead of Jacobi transform, we are now in the setting of Jacobi polynomial expansion of $f \in L^2(A, w(r)dr)$ given by

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The vanishing of f near 0 guarantees that $\sum_{c_\lambda \in E} c_\lambda^m \, d_\lambda \hat{f}(\lambda) = 0$, whereas

$$\int_{\mathbb{R}} t^m d\mu_f(t) = \sum_{c_\lambda \in E} c_\lambda^m \, d_\lambda |\hat{f}(\lambda)|$$

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Theorem: Let θ be a positive decreasing function on $(0, \infty)$ vanishing at infinity that satisfies $\int_1^\infty \theta(t) t^{-1} dt = \infty$. Then for any function $f \in L^2(G)$ vanishing on a nonempty open set $V \subset G$ the estimates

$$\sup_{g \in V} |f * \varphi_\lambda(g)| \leq C e^{-\sqrt{c_\lambda}\,\theta(\sqrt{c_\lambda})},$$

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As the spherical functions satisfy the identity $\varphi_\lambda * \varphi_\lambda = \varphi_\lambda$, we can replace the hypothesis by $\|f * \varphi_\lambda\|_2 \leq C e^{-\sqrt{c_\lambda}\, \theta(\sqrt{c_\lambda})}$ or even by

$$\sup_{g \in V^c} |f * \varphi_\lambda(g)| \leq C e^{-\sqrt{c_\lambda}\, \theta(\sqrt{c_\lambda})}.$$

Hermite and special Hermite operators

In order to prove spectral versions of Ingham's theorem for Hermite and special Hermite operators, it is instructive to study the sublaplacian $\mathcal{L}$ on the Heisenberg group $\mathbb{H}^n$. The group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ is quipped with the group law

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$$(z,t)(w,s) = (z+w, t+s+\frac{1}{2}\operatorname{Im}(z\cdot\bar{w}))$$

and the sublaplacian is given explicitly by

$$\mathcal{L} = -\Delta_{\mathbb{C}^n} - \frac{1}{4}|z|^2\frac{\partial^2}{\partial t^2} + \sum_{j=1}^{n}(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j})\frac{\partial}{\partial t}.$$

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The Hermite and special Hermite operators are related to $\mathcal{L}$ via

$$\mathcal{L}(f(z)e^{i\lambda t}) = e^{i\lambda t}L_\lambda f(z), \;\; \pi_\lambda(\mathcal{L}) = H(\lambda)$$

where $\pi_\lambda, 0 \neq \lambda \in \mathbb{R}$ are the Schrödinger representations of $\mathbb{H}^n$ on $L^2(\mathbb{R}^n)$.

Hermite and special Hermite operators

Written explicitly, $H(\lambda) = -\Delta + \lambda^2|x|^2$ and

$$L_\lambda = -\Delta_{\mathbb{C}^n} + \frac{1}{4}\lambda^2|z|^2 + i\lambda\sum_{j=1}^{n}(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}).$$

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The spectrum of both $H(\lambda)$ and L_λ consist of the points $(2k+n)|\lambda|, k \in \mathbb{N}$ and they have very explicit spectral decompositions; for example,

$$L_\lambda f(z) = (2\pi)^{-n}|\lambda|^n \sum_{k=0}^{\infty}(2k+n)|\lambda|\, f *_\lambda \varphi_k^\lambda(z)$$

where $*_\lambda$ stands for the twisted convolution

$$f *_\lambda g(z) = \int_{\mathbb{C}^n} f(z-w)g(w)e^{i\frac{\lambda}{2}\operatorname{Im}(z\cdot\bar{w})}dw$$

and $\varphi_k^\lambda(z)$ are the Laguerre functions of type $(n-1)$:

$$\varphi_k^\lambda(z) = L_k^{n-1}(\frac{1}{2}|\lambda||z|^2)e^{-\frac{1}{4}|\lambda||z|^2}.$$

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The spectrum of $\mathcal{L}$ consists of the rays $(2k+n)|\lambda|, k \in \mathbb{N}, 0 \neq \lambda \in \mathbb{R}$, and the spectral decomposition of $\mathcal{L}$ is given by

$$\mathcal{L}f(z,t) = (2\pi)^{-n-1} \int_{-\infty}^{\infty} e^{-i\lambda t} \Big(\sum_{k=0}^{\infty} (2k+n)|\lambda| \, f^\lambda *_\lambda \varphi_k^\lambda(z) \Big) |\lambda|^n d\lambda$$

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We say that f is radial if it radial in the z variable and by abusing the notation we write $f(z,t) = f(r,t), r = |z|$. The action of $\mathcal{L}$ on such radial functions is given by the operator $\mathcal{L}_0 = -\frac{\partial^2}{\partial r^2} - \frac{2n-1}{r}\frac{\partial}{\partial r} - r^2 \frac{\partial^2}{\partial t^2}$ and for each nonzero λ we set

$$\mathcal{L}_0^{\lambda} = -\frac{\partial^2}{\partial r^2} - \frac{2n-1}{r}\frac{\partial}{\partial r} + \lambda^2 r^2.$$

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The spectral decomposition of the operator $\mathcal{L}_0$ is then given by

$$\mathcal{L}_0 f(r,t) = (2\pi)^{-n-1} \int_{-\infty}^{\infty} e^{-i\lambda t} \Big(\sum_{k=0}^{\infty} (2k+n)|\lambda| \, R_k^\lambda(f) \varphi_k^\lambda(r) \Big) d\lambda$$

Hermite and special Hermite operators

In the above expansion, the coefficients $R_k^\lambda(f)$ are given by

$$R_k^\lambda(f) = |\lambda|^n \frac{k!(n-1)!}{(k+n-1)!} \int_0^\infty f^\lambda(r)\varphi_k^\lambda(r) r^{2n-1} dr.$$

Thus the spectral decomposition of $\mathcal{L}_0^\lambda$ is provided by the expansion in terms of the functions $e_k^\lambda(r) = \frac{k!(n-1)!}{(k+n-1)!}\varphi_k^\lambda(r)$.

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As before, using the theorem of de Jeu we can prove a version of Chernoff theorem for the operator $\mathcal{L}_0^\lambda$. This in turn will be used to prove Ingham type theorems for the Fourier transform on the Heisenberg group and for the spectral projections associated to $H(\lambda)$ and L_λ.

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We relate the spectral projections of $\mathcal{L}$ with that of $\mathcal{L}_0$ via spherical means on the Heisenberg group. These are averages taken over the orbits of the unitary group $U(n)$ which acts on $\mathbb{H}^n$ in the z variable.

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For each $r > 0$ let μ_r stand for the normalised surface measure on the sphere $|z| = r$ in $\mathbb{C}^n$ considered as a measure on $\mathbb{H}^n$ supported on

$$\{(z, 0) \in \mathbb{H}^n : |z| = r\}.$$

Hermite and special Hermite operators

The spherical means of a function f on $\mathbb{H}^n$ are then defined as the convolution

$$f * \mu_r(z,t) = \int_{\mathbb{H}^n} f(z-w, t - \frac{1}{2}\operatorname{Im}(z\cdot\bar{w}))d\mu_r(w).$$

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$$f^\lambda *_\lambda \mu_r(z) = (2\pi)^{-n}\Big(\sum_{k=0}^{\infty} f^\lambda *_\lambda \varphi_k^\lambda(z) e_k^\lambda(r)\Big)|\lambda|^n.$$

When f^λ vanishes on an open set $V^\lambda \subset \mathbb{C}^n$ it is clear that for any $z \in V^\lambda$, $f^\lambda *_\lambda \mu_r(z)$ vanishes for r near 0. Moreover,

$$\|(\mathcal{L}_0^\lambda)^m(f^\lambda *_\lambda \mu_r(z))\|_2^2 = c_n\Big(\sum_{k=0}^{\infty}((2k+n)|\lambda|)^{2m}|f^\lambda *_\lambda \varphi_k^\lambda(z)|^2\Big)|\lambda|^n.$$

Hermite and special Hermite operators

The spherical means of a function f on $\mathbb{H}^n$ are then defined as the convolution

$$f * \mu_r(z,t) = \int_{\mathbb{H}^n} f(z-w, t - \frac{1}{2}\operatorname{Im}(z\cdot\bar{w}))d\mu_r(w).$$

Considered as a function of r we can expand $f * \mu_r(z,t)$ in terms of the functions $e_k^\lambda(r)$ leading to

$$f^\lambda *_\lambda \mu_r(z) = (2\pi)^{-n}\Big(\sum_{k=0}^{\infty} f^\lambda *_\lambda \varphi_k^\lambda(z) e_k^\lambda(r)\Big)|\lambda|^n.$$

When f^λ vanishes on an open set $V^\lambda \subset \mathbb{C}^n$ it is clear that for any $z \in V^\lambda$, $f^\lambda *_\lambda \mu_r(z)$ vanishes for r near 0. Moreover,

$$\|(\mathcal{L}_0^\lambda)^m(f^\lambda *_\lambda \mu_r(z))\|_2^2 = c_n\Big(\sum_{k=0}^{\infty}((2k+n)|\lambda|)^{2m}|f^\lambda *_\lambda \varphi_k^\lambda(z)|^2\Big)|\lambda|^n.$$

Consequently, we have the following Ingham type theorem for the spectral projections associated to the sublaplacian $\mathcal{L}$.

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Theorem: Let θ be a positive decreasing function on $(0,\infty)$ vanishing at infinity that satisfies $\int_1^\infty \theta(t)t^{-1}dt = \infty$. Let $f \in L^2(\mathbb{H}^n)$ be such that for any nonzero λ, f^λ vanishes on a nonempty open set $V^\lambda \subset \mathbb{C}^n$. Then the estimates

$$\sup_{z\in V^\lambda} |f^\lambda *_\lambda \varphi_k^\lambda(z)| \leq Ce^{-\sqrt{(2k+n)|\lambda|}\ \theta(\sqrt{(2k+n)|\lambda|})},$$

cannot hold unless the function f vanishes identically.

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As $\varphi_k^\lambda *_\lambda \varphi_k^\lambda = (2\pi)^n |\lambda|^{-n} \varphi_k^\lambda$ and $\|g *_\lambda h\|_2 \leq |\lambda|^{-n} \|g\|_2 \|h\|_2$ we note that

$$(2\pi)^n |\lambda|^{-n} |f^\lambda *_\lambda \varphi_k^\lambda(z)| = |f^\lambda *_\lambda \varphi_k^\lambda *_\lambda \varphi_k^\lambda(z)|$$

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$$|\lambda|^n|f^\lambda *_\lambda \varphi_k^\lambda(z)|^2 \leq c_n\frac{(k+n-1)!}{k!(n-1)!}\|f^\lambda *_\lambda \varphi_k^\lambda\|_2^2.$$

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Therefore, the hypothesis in the theorem can be stated in terms of $\|f^\lambda *_\lambda \varphi_k^\lambda\|_2$.

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We also have a version of Ingham's theorem for the group Fourier transform on the Heisenberg group. For $f \in L^1(\mathbb{H}^n)$ and $0 \neq \lambda \in \mathbb{R}$ the Fourier transform $\hat{f}(\lambda)$ is a bounded linear operator on $L^2(\mathbb{R}^n)$.

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Theorem: Let $\Theta : \mathbb{R} \to [0, \infty)$ be an even, decreasing function with $\lim_{|\lambda|\to\infty} \Theta(\lambda) = 0$ and $I = \int_1^\infty \Theta(\lambda)\lambda^{-1} d\lambda = \infty$. Suppose the Fourier transform of $f \in L^1(\mathbb{H}^n)$ satisfies

$$\hat{f}(\lambda)^* \hat{f}(\lambda) \leq e^{-\Theta(\sqrt{H(\lambda)})\sqrt{H(\lambda)}}, \ \lambda \neq 0.$$

If f^λ vanishes on a non-empty open set V^λ for every λ, then $f = 0$ *a.e.*

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Spectral versions of Ingham type theorem have been proved for Hermite and special Hermite operators. Just for the sake of completeness let me state the result for $H = -\Delta + |x|^2$.

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Spectral versions of Ingham type theorem have been proved for Hermite and special Hermite operators. Just for the sake of completeness let me state the result for $H = -\Delta + |x|^2$. We write the spectral decomposition of H as

$$Hf(x) = \sum_{k=0}^{\infty} (2k+n) P_k f(x), \ P_k f(x) = \sum_{|\alpha|=k} (f, \Phi_\alpha)\Phi_\alpha(x)$$

where $\Phi_\alpha, \alpha \in \mathbb{N}^n$ are the normalised Hermite functions on $\mathbb{R}^n$.

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$$|P_k f(\xi)| \leq C_\xi e^{-\sqrt{2k+n}\,\theta(\sqrt{2k+n})},$$

cannot hold for all $\xi \in V$ unless the function vanishes identically.

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In proving Ingham's theorem in any setting, a crucial role is played by the existence of compactly supported functions whose Fourier transforms/spectral projections have a prescribed decay. For example in the original work of Ingham, he constructed compactly supported f on $\mathbb{R}$ whose Fourier transform satisfies

$$|\hat{f}(y)| \leq e^{-|y|\theta(|y|)}, \quad \int_1^\infty \theta(t)t^{-1}dt < \infty.$$

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For symmetric spaces (both compact and non compact), existence of such functions can be deduced using the Euclidean result. However, in the case of Heisenberg group Fourier transform, the construction is non trivial. For Hermite and special Hermite operators, one can construct examples in terms of the functions constructed on the Heisenberg group.

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Thanks for your attention