Dispersive Estimates for Schrödinger Equations

Ricardo Weder
weder@unam.mx
Universidad Nacional Autónoma de México

Seminar on Analysis, Differential Equations and Mathematical Physics.
Regional Mathematical Center of the Southern Federal University
Rostov-on-Don, Russia, October 29, 2020

Let us first consider the simple example of the free Schrödinger equation in $\mathbb{R}^n$,

$$i\frac{\partial}{\partial t}\varphi(t,x) = H_0\varphi(t,x),\ \varphi(0,x) = \psi(x), \quad H_0 := -\Delta.$$

By an elementary calculation, using the Fourier transform, we obtain that,

$$\varphi(t,x) = e^{-itH_0}\psi(x) = \frac{1}{(4\pi i t)^{\frac{n}{2}}} \int e^{\frac{i|x-y|^2}{4t}}\, \psi(y)\, dy.$$

From this explicit formula we obtain the $L^1 - L^\infty$ estimate,

$$\|\varphi(t,\cdot)\|_{L^\infty} \leq \frac{1}{(4\pi|t|)^{\frac{n}{2}}} \|\psi\|_{L^1}.$$

The proof of this estimate is elementary, and yet, it is a deep result.
Recall that the Schrödinger equation is invariant under time reversal: changing t to $-t$ and taking complex conjugate.
On spite of this, if for some time, say $t = 0$, the solution is integrable, then, it is bounded for all other times, and it goes uniformly to zero, in L^∞ norm, as $t \to \pm\infty$, at the rate, $1/|t|^{n/2}$.

This expresses in a quantitative way the wave packet spreading that we learn in the undergraduate quantum mechanics classes: as the wave packet propagates it spreads, and as the L^2 norm is constant is has to go to zero in pointwise sense

The following simple example illustrates the importance of this dispersive estimate. Consider the nonlinear Schrödinger equation,

$$i\frac{\partial}{\partial t}\varphi(t,x) = -\frac{\partial^2}{\partial x^2}\varphi(t,x) + \lambda|\varphi(t,x)|^p \frac{\varphi(t,x)}{|\varphi(t,x)|}, t, x \in \mathbb{R}, p \geq 5,$$

$\varphi(0,t) = \psi(x) \in H_1$.
where λ is a coupling constant.
Suppose that $\|\psi\|_{H_1} < \delta$ for δ small.Then, as the initial data is small for small times the nonlinear part of the equation is very small and the solution is dominated by the linear part. But, by the $L^1 - L^\infty$ estimate, as time increases the solution decreases in pointwise sense ,and the nonlinear part becomes even smaller. Hence, the solution is dominated for all times by the linear part, and then, there is no blowup, the solution exists globally in time, and moreover for large times it is asymptotic to a solution of the linear equation.

That is to say, for small initial data the Cauchy problem has a unique solution, global in time, and scattering holds, the wave and scattering operators exist.

This argument is basically a proof of the claim that I made. One just just to write an appropriate fixed point argument, and add the necessary details.

This simple example illustrates the importance of dispersive estimates, like the $L^1 - L^\infty$ estimate, the related $L^p - L^{p'}$, estimate, the Strichartz estimates, and other dispersive estimates, in spectral theory and in nonlinear analysis. Actually, this currently a very active area of research.

These comments serve the purpose of motivating the topic of this talk, that is, of course, dispersive estimates for Schrödinger equations with potentials.

Consider the interacting Schrödinger equation,

$$i\frac{\partial}{\partial t}\varphi(t,x) = H\varphi(t,x), t \in \mathbb{R}, x \in \mathbb{R}^n, \qquad H := -\Delta + V(x).$$

The appropriate $L^1 - L^\infty$ estimate in this case would be,

$$\left\| e^{-itH} P_{\mathrm{ac}}(H) \right\|_{\mathcal{B}(L^1, L^\infty)} \leq \frac{C}{t^{n/2}},$$

where $P_{\mathrm{ac}}(H)$ is the projector onto the absolutely continuos subspace of H.

The first result on this problem is from 1991.

[16] J.L. Journé, A. Soffer and C.D. Sogge, 1991, three or more dimensions.

Theorem. [16] Journé, Soffer, and Sogge, 1991.

Suppose that $n \geq 3$, that,

$$\langle x \rangle^{\alpha} V(x) : H_{\eta} \to H_{\eta}, \qquad , \eta > 0, \alpha > n+4,$$

$$\hat{V} := \mathcal{F} V \in L^1.$$

Then, the $L^p - L^{p'}$ estimate holds,

$$\left\| e^{-itH} P_{\mathrm{ac}}(H) \right\|_{\mathcal{B}(L^p, L^{p'})} \leq \frac{C}{t^{n(\frac{1}{p} - \frac{1}{p'})}},$$

$1 \leq p \leq 2, \frac{1}{p} + \frac{1}{p'} = 1.$

The motivation of Journé Soffer and Sogge was direct low-energy scattering for nonlinear Schrödinger equations

[31] R. W., 2000, the case of the line.

$$H = -\frac{d}{dx} + V(x), \text{in } L^2(\mathbb{R}).$$

We say that V belongs to L^1_γ if,

$$\int_{\mathbb{R}} (1 + |x|)^\gamma \, |V(x)| \, dx < \infty.$$

Theorem.[31] R.W., 2000 *Suppose that* $V \in L^1_\gamma$, *where in the generic case* $\gamma > \frac{3}{2}$ *and in the exceptional case* $\gamma > \frac{5}{2}$. *Then, for* $1 \leq p \leq 2$, *and* $\frac{1}{p} + \frac{1}{p'} = 1$, *the* $L^p - L^{p'}$ *estimate holds,*

$$\left\| e^{-itH} P_{\mathrm{ac}}(H) \right\|_{\mathcal{B}(L^p, L^{p'})} \leq \frac{C}{t^{\left(\frac{1}{p} - \frac{1}{p'}\right)}}.$$

Recall that the potential is exceptional if the zero energy Schrödinger equation has a bounded solution and that it is generic otherwise. If the potential is exceptional there is a zero energy resonance or a half-bound state.
The motivation of R. W. was direct and inverse low-energy scattering of nonlinear equations and also center manifolds, see [33] R. W. 2000.

[34] R. W., 2003, the case of the half line.

$$H = -\frac{d^2}{dx^2} + V(x), \text{ in } L^2(\mathbf{R}^+),$$

with Dirichlet boundary condition , $\psi(0) = 0$.

Theorem, [34] R. W., 2003 *Suppose that V satisfies*

$$\int_0^\infty x\,|V(x)|\,dx < \infty.$$

Then, for $1 \leq p \leq 2$, *and* $\frac{1}{p} + \frac{1}{p'} = 1$, *the* $L^p - L^{p'}$ *estimate holds,*

$$\left\| e^{-itH} P_{\mathrm{ac}}(H) \right\|_{\mathcal{B}(L^p, L^{p'})} \leq \frac{C}{t^{\left(\frac{1}{p} - \frac{1}{p'}\right)}}.$$

It is in this theorem where the $L^p - L^{p'}$ estimate was proven by the first time under the optimal decay condition on the potential, namely that $xV(x)$ is integrable at infinity, and where it was discovered that the $L^p - L^{p'}$ estimate holds under the optimal decay condition whether there is a zero-energy resonance or not. The fact that we do not need that the potential is locally integrable up to zero is specific to the Dirichlet boundary condition.

In his book,
[5] J. Bourgain, *Global Solutions of Nonlinear Schrödinger Equations,* Colloquium Publications **46**, A.M.S., Providence R. I., 1999,
J. Bourgain proposed as an open research problem to prove the $L^p - L^{p'}$ estimates in one and two dimensions. The results of R. W. 2000 where obtained independently of Bourgain's proposal.
The motivation of J. Bourgain was the study of the properties of the set where the solutions to the nonlinear Schrödinger equations blowup.
The fact that J. Bourgain posed as an open problem the proof of $L^p - L^{p'}$ estimates, played an important role in attracting the interest of mathematicians into dispersive estimates.

The sequel

After the early results that I described, the problem of dispersive estimates became a very active area of research, both in one dimension and in higher dimensions. Also other equations like the Klein-Gordon and the Dirac equation, and equations with magnetic field where considered. Furthermore, discrete equations, like discrete Schrödinger and Dirac equations, and quantum walks were studied. Besides the original motivations, there is currently a great deal of activity in connection with the stability of solitons.

Reviews of the literature are given in,
[28] W. Schlag, 2007.
[9] L. Fanelli, 2008.

The proofs in one dimension and in higher dimensions are fundamentally different. Since I will be mainly concerned with the one dimensional case, below I comment only in the literature in one dimension

The $L^p - L^{p'}$ estimate for the Schrödinger equation on the line.

[10] M. Goldberg and W. Schlag, 2004. $V \in L_1^1$ in the generic case and $V \in L_2^1$ in the exceptional case.

[6] P. D'Ancona, and S. Selberg, 2012. A potential in L_2^1 plus a step potential.

[7] I. E. Egorova, E. A. Kopylova, V. A. Marchenko, and G. Teschl, 2016. $V \in L_1^1$ both in the generic and the exceptional cases.
This result has the optimal L_1^1 condition as in [34] R. W, 2003, in the half-line.

The $L^p - L^{p'}$ estimate for the spherical Schrödinger equation.

[14] M. Holzleitner, A. Kostenko and G. Teschl, 2016.
[19] A. Kostenko, G. Teschl and J. Toloza, 2016.
[15] M. Holzleitner, A. Kostenko and G. Teschl, 2018.

The $L^p - L^{p'}$ estimate for the discrete Schrödinger equation.

[29] A. Stefanov, and P. G. Kevrekidis, 2007.
[8] I. Egorova, and E. A. Kopylova, and G. Teschl, 2015.
[4] D. Bambusi, and Z. Zhao, 2020.

The $L^p - L^{p'}$ estimate for the discrete Dirac equation.
[17] E. A. Kopylova and G. Teschl, 2017.
[18] E. A. Kopylova, and G. Teschl, 2020.

The $L^p - L^{p'}$ estimates for the Klein-Gordon equation.
[32] R. W., 2000.
[26] O. Prill, 2014.
[7] I. E. Egorova, E. A. Kopylova, V. A. Marchenko and G. Teschl, 2016.

The $L^p - L^{p'}$ estimate for the Schrödinger equation on trees.
[3] K. Ammari, and M. Sabri, 2020.

The $L^p - L^{p'}$ estimate for the Schrödinger equation on quantum walks.
[22] M. Maeda, H. Sasaki, E. Segawa, A. Suzuki, and K. Suzuki, 2018.
[23] M. Maeda, H. Sasaki, E. Segawa, A. Suzuki, and K. Suzuki, 2020.

Matrix Schrödinger equation on the Half-Line

Let us consider the matrix Schrödinger equation on the half-line with general selfadjoint boundary condition

$$\begin{cases} i\partial_t u(t,x) = \left(-\partial_x^2 + V(x)\right) u(t,x), \; t \in \mathbb{R}, \; x \in \mathbb{R}^+, \\ u(0,x) = u_0(x), \; x \in \mathbb{R}^+, \\ -B^\dagger u(t,0) + A^\dagger (\partial_x u)(t,0) = 0, \end{cases}$$

where $\mathbb{R}^+ := (0,+\infty)$, $u(t,x)$ is a function from $\mathbb{R} \times \mathbb{R}^+$ into $\mathbb{C}^n$, A, B are constant $n \times n$ matrices, the potential V is a $n \times n$ selfadjoint matrix-valued function of x, i.e.

$$V(x) = V^\dagger(x), \; x \in \mathbb{R}^+,$$

where the dagger denotes the matrix adjoint. We suppose that V is L^1_1, i.e. that it is a Lebesgue measurable matrix-valued function and,

$$\int_{\mathbb{R}^+} (1+x)\,|V(x)|\,dx < \infty.$$

We find it convenient to state the general selfadjoint boundary condition requiring that the matrices A and B satisfy

$$B^{\dagger}A = A^{\dagger}B,$$

and

$$A^{\dagger}A + B^{\dagger}B > 0.$$

We denote by

$$H = H_{A,B,V} := -\frac{d^2}{dx^2} + V(x),$$

the selfadjoint realization in $L^2(\mathbb{R}^+)$ with the general selfadjoint boundary condition.

Theorem. [25] I. Naumkin, and R. W., 2020.
Suppose that the potential V is selfadjoint and belogns to L^1_1. Then, for any $p \in [1,2]$ and p' such that $1/p + 1/p' = 1$, the $L^p - L^{p'}$ estimate holds,

$$\left\| e^{-itH} P_{\rm ac}(H) \right\|_{\mathcal{B}\left(L^p, L^{p'}\right)} \leq \frac{C}{|t|^{1/p - 1/2}}.$$

For a star graph with the Kirchoff boundary condition and a potential in $L^1_\gamma, \gamma > 5/2$, the $L^p - L^{p'}$ estimate, was proven in [24] F. Ali Mehmeti, K. Ammari and S. Nicaise, 2015.

In the case of star graphs with potential identically zero, and with general boundary conditions, the $L^p - L^{p'}$ estimate, was obtained by [11] A. Grecu, and L. I. Ignat, 2019.

Corollary. Strichartz estimate. [25] I. Naumkin, R.W., 2020. Let (q,r) be an admissible pair, that is, $2/q = 1/2 - 1/r$ and $2 \leq r \leq \infty$. Then, for every $\varphi \in L^2(\mathbb{R}^+)$, the function $t \to e^{-itH} P_{\mathrm{ac}}(H)\varphi$ belongs to $L^q(\mathbb{R}, L^r(\mathbb{R}^+)) \cap C\left(\mathbb{R}, L^2(\mathbb{R}^+)\right)$,and

$$\left\| e^{-itH} P_{\mathrm{ac}}(H)\varphi \right\|_{L^q(\mathbb{R},L^r(\mathbb{R}^+))} \leq C \|\varphi\|_{L^2(\mathbb{R}^+)}, \qquad \varphi \in L^2(\mathbb{R}^+).$$

Moreover, let $I \subset \mathbb{R}$ be an interval. For an admissible pair (γ, ρ), let $f \in L^{\gamma'}\left(I, L^{\rho'}(\mathbb{R}^+)\right)$, where $1/\gamma + 1/\gamma' = 1$ and $1/\rho + 1/\rho' = 1$. Then, for $t_0 \in \bar{I}$, the function

$$t \to \Phi_f(t) = \int_{t_0}^{t} e^{-i(t-s)H} P_{\mathrm{ac}}(H) f(s)\, ds, \ \ t \in I,$$

belongs to $L^q(I, L^r(\mathbb{R}^+)) \cap C\left(\bar{I}, L^2(\mathbb{R}^+)\right)$ and for and I independent constant C

$$\|\Phi_f\|_{L^q(I,L^r(\mathbb{R}^+))} \leq C \|f\|_{L^{\gamma'}(I,L^{\rho'}(\mathbb{R}^+))}, \text{ for every } f \in L^{\gamma'}\left(I, L^{\rho'}(\mathbb{R}^+)\right).$$

The result of [25] I. Naumkin, and R. W., 2020 shows that the $L^p - L^{p'}$ estimate holds for general matrix Schrödinger operators on the half-line for potentials in L^1_1, whether there is a zero-energy resonance or not, as was first proven by [34] R. W., 2003, in the scalar case.

We are in the generic case if the Jost matrix is invertible at zero energy and we are in the exceptional case if the Jost matrix is not invertible at zero energy. In the exceptional case there is a zero-energy resonance (or half-bound state), and in the generic case there is no zero-energy resonance.

The matrix Schrödinger equation on the full-line

A $2n \times 2n$ matrix Schrödinger equation on the half-line is unitarily equivalent to a $n \times n$ matrix Schrödinger equation on the full-line with a point interaction at $x = 0$.

We define the unitary operator **U** from $L^2\left(\mathbb{R}^+; \mathbb{C}^{2n}\right)$ onto $L^2\left(\mathbb{R}; \mathbb{C}^n\right)$ by

$$\phi(x) = \mathbf{U}\psi(x) := \begin{cases} \psi_1(x), & x \geq 0, \\ \psi_2(-x), & x < 0, \end{cases}$$

for a vector-valued function $\psi = (\psi_1, \psi_2)^T$, where $\psi_j \in L^2\left(\mathbb{R}^+; \mathbb{C}^n\right), j = 1, 2$. Let the potential in the half-line Schrödinger equation be the diagonal matrix

$$V(x) := \mathrm{diag}\{V_1(x), V_2(x)\},$$

where $V_j, j = 1, 2$ are selfadjoint $n \times n$ matrix-valued functions that satisfy $V_j \in L^1_1(\mathbb{R}^+)$.

Under **U** the Hamiltonian H is transformed into the following Hamiltonian in the full-line,

$$H_{\mathbb{R}} := \mathbf{U}\, H \mathbf{U}^{\dagger}, \quad D[H_{\mathbb{R}}] := \{\phi \in L^2(\mathbb{R}; \mathbb{C}^n) : \mathbf{U}^{\dagger}\phi \in D[H]\}.$$

The operator $H_{\mathbb{R}}$ is a selfadjoint realization in $L^2(\mathbb{R}; \mathbb{C}^n)$ of the formal differential operator $-\partial_x^2 + Q(x)$ where,

$$Q(x) = \begin{cases} V_1(x), & x \geq 0, \\ V_2(-x), & x < 0. \end{cases}$$

Let us write the $2n \times 2n$ matrices A, B as follows,

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

with $A_j, B_j, j = 1, 2$, being $n \times 2n$ matrices. The functions in the domain of $H_{\mathbb{R}}$ satisfy the following transmission condition.

$$-B_1^{\dagger}\phi(0+) - B_2^{\dagger}\phi(0-) + A_1^{\dagger}(\partial_x\phi)(0+) - A_2^{\dagger}(\partial_x\phi)(0-) = 0.$$

Then, $u(t,x)$ is a solution of the half-line Schrödinger equation if and only if $v(t,x) := \mathbf{U}u(t,x)$ is a solution of the following full-line matrix Schrödinger equation,

$$\begin{cases} i\partial_t v(t,x) = \left(-\partial_x^2 + Q(x)\right) v(t,x), \ t \in \mathbb{R}, \ x \in \mathbb{R}, \\ v(0,x) = v_0(x) := \mathbf{U}u_0(x), x \in \mathbb{R}, \\ -B_1^\dagger v(t,0+) - B_2^\dagger v(t,0-) + A_1^\dagger (\partial_x v)(t,0+) - A_2^\dagger (\partial_x v)(t,0-) = 0. \end{cases}$$

For example, let us take,

$$A = \begin{bmatrix} 0_n & I_n \\ 0_n & I_n \end{bmatrix}, \quad B = \begin{bmatrix} -I_n & \Lambda \\ I_n & 0_n \end{bmatrix},$$

where Λ is a selfadjoint $n \times n$ matrix. In this case, the transmission condition is given by,

$$v(t,0+) = v(t,0-) = v(t,0), \quad (\partial_x v)(t,0+) - (\partial_x v)(t,0-) = \Lambda v(t,0).$$

This corresponds to a Dirac delta point interaction at $x = 0$ with coupling matrix Λ. If $\Lambda = 0$, $v(t,x)$ and $(\partial_x v)(t,x)$ are continuous at $x = 0$ and we get the matrix Schrödinger equation without a point interaction.

Using this unitary transformation we obtain the following results.

Theorem [25] I. Naumkin, R. Weder, 2020.

Suppose that $Q(x)$, $x \in \mathbb{R}$, is a $n \times n$ selfadjoint matrix-valued function such that $Q \in L^1_1(\mathbb{R}; \mathbb{C}^n)$. Then, for any $p \in [1, 2]$ and p' such that $1/p + 1/p' = 1$, the $L^p - L^{p'}$ estimate holds,

$$\left\| e^{-itH_{\mathbb{R}}} P_{\mathrm{ac}}(H_{\mathbb{R}}) \right\|_{\mathcal{B}\left(L^p(\mathbb{R};\mathbb{C}^n), L^{p'}(\mathbb{R};\mathbb{C}^n)\right)} \leq \frac{C}{|t|^{1/p-1/2}}.$$

I am not aware of any result on the $L^p - L^{p'}$ estimate for matrix Schrödinger equations on the full-line, even without point interaction. In the scalar case the result with point interaction is new.

Corollary. Strichartz estimate.[25] I. Naumkin, R.W., 2020.

Let (q, r) be an admissible pair, that is, $2/q = 1/2 - 1/r$ and $2 \leq r \leq \infty$. Then, for every $\varphi \in L^2(\mathbb{R})$, the function $t \to e^{-itH} P_{\rm ac}(H_{\mathbb{R}})\varphi$ belongs to $L^q(\mathbb{R}, L^r(\mathbb{R})) \cap C\left(\mathbb{R}, L^2(\mathbb{R})\right)$, and

$$\left\| e^{-itH} P_{\rm ac}(H_{\mathbb{R}})\varphi \right\|_{L^q(\mathbb{R}, L^r(\mathbb{R}))} \leq C \|\varphi\|_{L^2(\mathbb{R})}, \qquad \varphi \in L^2(\mathbb{R}).$$

Moreover, let $I \subset \mathbb{R}$ be an interval. For an admissible pair (γ, ρ), let $f \in L^{\gamma'}\left(I, L^{\rho'}(\mathbb{R})\right)$, where $1/\gamma + 1/\gamma' = 1$ and $1/\rho + 1/\rho' = 1$. Then, for $t_0 \in \overline{I}$, the function

$$t \to \Phi_f(t) = \int_{t_0}^{t} e^{-i(t-s)H} P_{\rm ac}(H) f(s)\, ds, \ \ t \in I,$$

belongs to $L^q(I, L^r(\mathbb{R})) \cap C\left(\overline{I}, L^2(\mathbb{R})\right)$ and for and I independent constant C

$$\|\Phi_f\|_{L^q(I, L^r(\mathbb{R}^+))} \leq C \|f\|_{L^{\gamma'}(I, L^{\rho'}(\mathbb{R}))}, \text{ for every } f \in L^{\gamma'}\left(I, L^{\rho'}(\mathbb{R})\right).$$

The proofs

The method to prove the $L^1 - L^\infty$ estimate is to give a formula for the integral kernel of $e^{-itH} P_{\rm ac}(H)$ in terms of the physical solution. The physical solution is defined using the Jost solution and the scattering matrix. Furthermore, Wiener algebra methods are used, and for this purpose precise low- and high-energy estimates are crucial.

This is basically the method that has been used to prove the $L^1 - L^\infty$ estimate in one dimension since the work of [31] R. W. 2000, [34] R.W. 2003.

We state results that we need and that are in,

[2] T. Aktosun and R. W., *Direct and Inverse Scattering for the Matrix Schrödinger Equation,* Applied Mathematical Sciences **203** , Springer Verlag, New York, published in May 2020.

Consider the stationary matrix Schrödinger equation on the half-line

$$-\psi'' + V(x)\,\psi = k^2\psi,\ x \in \mathbb{R}^+,$$

$V(x)$ is selfadjoint and satisfies,

$$V \in L^1\left(\mathbb{R}^+\right).$$

The wavefunction $\psi(k,x)$ may be either a $n \times n$ matrix-valued function or it may be a column vector with n components.

Recall that the more general selfadjoint boundary condition can be expressed in terms of two matrices A and B as

$$-B^{\dagger}\psi(0) + A^{\dagger}\psi'(0) = 0,$$

where A and B satisfy

$$B^{\dagger}A = A^{\dagger}B,$$

$$A^{\dagger}A + B^{\dagger}B > 0.$$

For other, equivalent, characterizations of the boundary conditions see [20] V. Kostrykin and R. Schrader, 1999, [13] M. S. Harmer,2004, and [27] F. S. Rofe-Beketov, and A. M. Kholkin, 2005.

Let $\tilde{A}$ and $\tilde{B}$, given by

$$\tilde{A} = -\,\mathrm{diag}[\sin\theta_1, ..., \sin\theta_n], \;\; \tilde{B} = \mathrm{diag}[\cos\theta_1, ..., \cos\theta_n],$$

with real parameters $\theta_j \in (0, \pi]$. For the matrices $\tilde{A}$, $\tilde{B}$, the boundary conditions are given by

$$\cos\theta_j \psi_j(0) + \sin\theta_j \psi_j'(0) = 0, \quad j = 1, 2, ..., .,$$

The special case $\theta_j = \pi$ corresponds to the Dirichlet boundary condition and the case $\theta_j = \pi/2$ corresponds to the Neumann boundary condition. In general, there are $n_{\mathrm{N}} \leq n$ values with $\theta_j = \pi/2$ and $n_{\mathrm{D}} \leq n$ values with $\theta_j = \pi$, and hence there are n_{M} remaining values, with $n_{\mathrm{M}} = n - n_{\mathrm{N}} - n_{\mathrm{D}}$ such that those θ_j-values lie in the interval $(0, \pi/2)$ or $(\pi/2, \pi)$, i.e., they correspond to mixed boundary conditions.

It is proven in [2] T. Aktosun, and R. W, 2020, that for any pair of matrices (A, B) that characterize the boundary conditions, there is a pair of matrices $(\tilde{A}, \tilde{B})$, a unitary matrix M and two invertible matrices T_1, T_2 such

$$A = M\tilde{A}T_1 M^\dagger T_2, \quad B = M\tilde{B}T_1 M^\dagger T_2,$$

and, further,

$$H_{A,B,V} = MH_{\tilde{A},\tilde{B},M^\dagger VM}M^\dagger.$$

This shows that the case of general boundary conditions is unitarily equivalent to the case of diagonal boundary matrices, where the boundary conditions are just the usual ones of the scalar case.

This is technically very useful. For example, in the case where the potential is zero, the proof of the $L^1 - L^\infty$ estimate follows immediately from the explicit representation of the unitary propagator in the scalar case with Dirichlet, Neumann and mixed boundary conditions.

The representation with diagonal boundary matrices it is not only technically convenient, but it is essential to understand the low- and high- energy limit of the Jost and the scattering matrices, because they depend explicitly in the number of Dirichlet boundary conditions, n_{D} in the diagonal representation.

[1] Z. S. Agranovich and V. A. Marchenko, 1963.
For each fixed $k \in \overline{\mathbb{C}^+} \setminus \{0\}$ there exists a unique $n \times n$ matrix-valued Jost solution $f(k,x)$ satisfying the asymptotic condition as $x \to \infty$,

$$f(k,x) = e^{ikx}(I + o(1)), \quad f'(k,x) = e^{ikx}[ik\,I + o(1)].$$

The Jost matrix $J(k)$ is the $n \times n$ matrix-valued function of k,

$$J(k) = f(-k^*,0)^\dagger B - f'(-k^*,0)^\dagger A, \;\; k \in \overline{\mathbb{C}^+}.$$

The Jost matrix $J(k)$ is analytic for $k \in \mathbb{C}^+$, continuous for $k \in \overline{\mathbb{C}^+} \setminus \{0\}$ and invertible for $k \in \mathbb{R} \setminus \{0\}$. If furthermore, the potential belongs to L^1_1 then, the Jost matrix is continuos for $k \in \overline{\mathbb{C}^+}$.

From the Jost matrix $J(k)$ we construct the scattering matrix $S(k)$, which is a $n \times n$ matrix-valued function of k given by

$$S(k) = -J(-k)\, J(k)^{-1}, \; k \in \mathbb{R}.$$

In the exceptional case where $J(0)$ is not invertible the scattering matrix is defined as above only for $k \neq 0$. However, for potentials in L^1_1 the limit $S(0) := \lim_{k \to 0} S(k)$ exists in the exceptional case.

In terms of the Jost solution $f(k,x)$ and the scattering matrix $S(k)$ we construct the physical solution

$$\Psi(k,x) = f(-k,x) + f(k,x)\, S(k), k \in \mathbb{R}.$$

The physical solution Ψ is the basis to construct the generalized Fourier maps for the absolutely continuous subspace of H. We define,

$$\left(\mathbf{F}^{\pm}\psi\right)(k) = \sqrt{\frac{1}{2\pi}} \int_0^{\infty} \left(\Psi\left(\mp k, x\right)\right)^{\dagger} \psi(x)\, dx,$$

for $\psi \in L^2 \cap L^1$.
For any Borel set O let $E(O)$ be the spectral projector of H for O. Then,

$$\left\|\mathbf{F}^{\pm}\psi\right\|_{L^2} = \left\|E(\mathbb{R}^+)\psi\right\|_{L^2}.$$

Thus, $\mathbf{F}^{\pm}$ extend to bounded operators on L^2 that we also denote by $\mathbf{F}^{\pm}$.

H has no positive bound states, and the negative spectrum of H consists of isolated bound states of multiplicity smaller or equal than n, that can accumulate only at zero.
H has no singular continuous spectrum and its absolutely continuous spectrum is given by $[0, \infty)$.
The generalized Fourier maps $\mathbf{F}^{\pm}$ are partially isometric with initial subspace $\mathcal{H}_{ac}(H)$ and final subspace L^2.

$$\mathbf{F}^{\pm} H \left(\mathbf{F}^{\pm}\right)^{\dagger} = k^2.$$

If $V \in L^1_1$, there is no bound state at $k = 0$ and the number of bounded states of H is finite.

We observe that in particular we have,

$$e^{-itH} P_{\mathrm{ac}}(H) = \left(\mathbf{F}^{\pm}\right)^{\dagger} e^{-itk^2} \mathbf{F}^{\pm}.$$

Using the definition of the physical solution Ψ, we get

$$e^{-itH} P_{\mathrm{c}} \psi = (2\pi)^{-1} \int_0^{\infty} \mathcal{T}(x, y) \, \psi(y) \, dy,$$

where

$$\mathcal{T}(x, y) = \int_{-\infty}^{\infty} e^{-itk^2} \Big(f(-k, x) \left(f(-k, y) \right)^{\dagger} + f(k, x) \, S(k) \left(f(-k, y) \right)^{\dagger} \Big) \, dk.$$

This equation is the starting point of the proof of the $L^1 - L^{\infty}$ estimate.

The Jost solution $f(k,x)$ has the representation ([1] Z. S. Agranovich and V. A. Marchenko, 1963.)

$$f(k,x) = e^{ikx} I + \int_x^\infty e^{iky} K(x,y)\, dy,$$

where the matrix $K(x,y)$ satisfies,

$$K(x,y) = 0, \ \ y < x, x, y \in [0,\infty),$$
$$|K(x,y)| \leq \frac{1}{2} e^{\sigma_1(x)} \sigma\left(\frac{x+y}{2}\right), \ \ x, y \in [0,\infty).$$

with

$$\sigma(x) = \int_x^\infty |V(y)|\, dy, \ \ \sigma_1(x) = \int_x^\infty y\, |V(y)|\, dy, \ \ x \geq 0.$$

Using this representation of the Jost solution we decompose $\mathcal{T}(x, y)$ into six terms ([25] I. Naumkin, R. W., 2020) ,

$$\mathcal{T}(x, y) = \sum_{j=0}^{6} \mathcal{T}_j(x, y).$$

We have to estimate each one of them.

$$\mathcal{T}_0 = \sqrt{\frac{\pi}{it}} \left(e^{i(x-y)^2/4t} + e^{i(x+y)^2/4t} S_\infty \right).$$

Let us concentrate in $\mathcal{T}_3$, that is given by,

$$\mathcal{T}_3 = \frac{2\pi}{\sqrt{4\pi it}} \int_{-\infty}^{\infty} e^{i(x+y-z)^2/4t} F_s(z)\, dz,$$

where F_s is the Fourier transform of, $S(k) - S_\infty$,

$$F_s(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [S(k) - S_\infty]\, e^{iky} dk, \quad y \in \mathbb{R},$$

with,

$$S_\infty := \lim_{k \to \pm\infty} S(k).$$

[25] I. Naumkin, and R.W., 2020 Suppose that the potential V is selfadjoint and that $V \in L^1_1$. Then,

$$F_s \in L^1(\mathbb{R}).$$

The key input for the proof of this theorem are precise low- and high-energy asymptotics for the Jost and the scattering matrices that are given in

[2] T. Aktosun and R. W., *Direct and Inverse Scattering for the Matrix Schrödinger Equation,* Applied Mathematical Sciences **203** , Springer Verlag, New York, published in May 2020.
We proceed to discuss these estimates.

Let μ be the geometric multiplicity of the zero eigenvalue of $J(0)$, and ν the algebraic multiplicity.

The Jordan canonical form of $J(0)$

$$\mathcal{S}^{-1} J(0)\, \mathcal{S} = \oplus_{\alpha=1}^{\kappa} J_{n_\alpha}(\lambda_\alpha),$$

where $J_{n_\alpha}(\lambda_\alpha)$ is the $n_\alpha \times n_\alpha$ Jordan block

$$J_{n_\alpha}(\lambda_\alpha) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \qquad \alpha = 1, \dots, \mu,$$

$$J_{n_\alpha}(\lambda_\alpha) = \begin{bmatrix} \lambda_\alpha & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_\alpha & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_\alpha & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda_\alpha \end{bmatrix}, \qquad \alpha = \mu + 1, \dots, \kappa.$$

Let us denote

$$\tilde{M} := \mathcal{S}^{-1} M \mathcal{S}$$

for any $n \times n$ matrix M.

$$P_2\tilde{J}(0)\,P_1 = \mathrm{diag}\{0_\mu, I_{\nu-\mu}, J_{n_\mu+1}(\lambda_{\mu+1}), \ldots, J_{n_\kappa}(\lambda_\kappa)\},$$

where 0_μ denotes the $\mu \times \mu$ zero matrix.

$$P_1 = \begin{bmatrix} \Pi_1 & 0 \\ 0 & I_{n-\nu} \end{bmatrix}, \quad P_2 = \begin{bmatrix} \Pi_2 & 0 \\ 0 & I_{n-\nu} \end{bmatrix},$$

for some permutation matrices Π_1 and Π_2.

Theorem T. Aktosun, and R. W., 2020.

Assume that V is selfadjoint and belongs to L^1_1. Then, as $k \to 0$ in $\overline{\mathbf{C}^+}$ the Jost matrix $J(k)$ has the behavior

$$J(k) = \mathcal{S} P_2^{-1} \begin{bmatrix} k\mathcal{A}_1 + o(k) & k\mathcal{B}_1\mathcal{A}_1 + o(k) \\ k\mathcal{C}_1 + o(k) & \mathcal{D}_0 + o(1) \end{bmatrix} P_1 \mathcal{S}^{-1},$$

the inverse Jost matrix $J(k)^{-1}$ has the behavior, as $k \to 0$ in $\overline{\mathbf{C}^+}$,

$$J(k)^{-1} = \mathcal{S} P_1 \begin{bmatrix} \frac{1}{k}\mathcal{A}_1^{-1}[I_\mu + o(1)] & -\mathcal{A}_1^{-1}\mathcal{B}_1\mathcal{D}_0^{-1} + o(1) \\ -\mathcal{D}_0^{-1}\mathcal{C}_1\mathcal{A}_1^{-1} + o(1) & \mathcal{D}_0^{-1} + o(1) \end{bmatrix} P_2 \mathcal{S}^{-1},$$

and the scattering matrix $S(k)$ is continuous at $k = 0$ and we have $S(k) = S(0) + o(1)$ as $k \to 0$ in $\mathbf{R}$ with

$$S(0) = \mathcal{S} P_2^{-1} \begin{bmatrix} I_\mu & 0 \\ 2\mathcal{C}_1 \mathcal{A}_1^{-1} & -I_{n-\mu} \end{bmatrix} P_2 \mathcal{S}^{-1},$$

where μ is the geometric multiplicity of the zero eigenvalue of the zero-energy Jost matrix $J(0)$.

Large-k behavior of $S(k)$.

We define,

$$Q_1 := \frac{1}{2}\int_0^\infty dy\, V(y), \quad Q_2(k) := \frac{1}{2}\int_0^\infty dy\, e^{2iky}\, V(y),$$

$$Q_3 := \frac{1}{4}\int_0^\infty dz \int_0^z dy\, V(z)\, V(y),$$

$$Q_4(k) := \frac{1}{4}\int_0^\infty dz \int_0^z dy\, e^{2ikz}\, V(z)\, V(y),$$

$$Q_5(k) := \frac{1}{4}\int_0^\infty dz \int_0^z dy\, e^{2iky}\, V(z)\, V(y),$$

$$Q_6(k) := \frac{1}{4}\int_0^\infty dz \int_0^z dy\, e^{2ik(z-y)}\, V(z)\, V(y).$$

Theorem T. Aktosun, and R.W., 2020.

Assume that V is selfadjoint and that, $V \in L^1(\mathbf{R}^+)$. Then,

$$S(k) = S_0(\infty) + \frac{G(k)}{ik} + O(1/k^2), \qquad k \to \pm\infty,$$

where

$$S_0(\infty) := M \,\mathrm{diag}\{I_{n_M}, -I_{n_D}, I_{n_N}\}\, M^\dagger,$$

is the high-energy limit for potential zero. M is a unitary matrix and $G(k)$ is the matrix defined as

$$G(k) := -2M\, Z_1\, M^\dagger + Q_1 S_0(\infty) + S_0(\infty)\, Q_1 +$$
$$S_0(\infty) Q_2(k) S_0(\infty) + Q_2(-k),$$

with

$$Z_1 := \mathrm{diag}\,\{\cot\theta_1, \ldots, \cot\theta_{n_M}, 0_{n_D}, 0_{n_N}\},$$

where, 0_j is the zero $j \times j$ matrix.

The L^p boundedness of the wave operators

K. Yajima introduced a different method to prove dispersive estimates in [36] K. Yajima, 1995. See [37] K. Yajima, 2020, for a review and recent results.

Denote $H_0 := -\Delta$, and $H := H_0 + V$ in $L^2(\mathbb{R}^n)$, $n \geq 1$. Assume that the wave operators,

$$W_\pm = \mathrm{s}\text{-}\lim_{t \to \pm\infty} e^{itH} e^{-itH_0},$$

exist and are complete. Then, by the intertwining relations,

$$f(H) P_{\mathrm{ac}}(H) = W_\pm^* f(H_0)\, W_\pm.$$

If the wave operators are bounded in $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, then, from the dispersive estimates for H_0 we obtain the dispersive estimates for H.

The boundedness in L^p of the wave operators is an interesting problem on its own, and it has been proved in many circumstances. However, it often fails in the presence of threshold singularities if $n \geq 2$.

In [30] R. W. ,1999, it has been proved that in the one-dimensional case the wave operators are not always bounded in L^1 and in L^∞

Theorem, R. W. 1999 Suppose that $V \in L^1_\gamma(\mathbb{R})$, with $\gamma > 3/2$ in the generic case and $\gamma > 5/2$ in the exceptional case. Then, the wave operators $W_\pm$ and $W^*_\pm$ are bounded in L^p, $1 < p < \infty$. Furthermore, $W_\pm$ are bounded from L^1 into L^1_{weak}, and $W^*_\pm$ are bounded from L^∞ into BMO. Moreover, in the exceptional case if $\lim_{x \to -\infty} f(0,x) = 1$, the $W_\pm$ and the $W^*_\pm$ are bounded in L^1 and in L^∞.

This is due to the presence of a term with the Hilbert transform in the low-energy asymptotics

Theorem. [2] T. Aktosun, R. W. 2020
Consider the Schrödinger operator with the general selfadjoint boundary condition and with the potential $V \in L^1_1((\mathbf{R}^+))$. *The number* $\mathcal{N}$ *of bound states (including multiplicities) is related to the argument of the determinant of the scattering matrix* $S(k)$ as

$$\arg[\det S(0^+)] - \arg[\det S(+\infty)] = \pi\,(2\mathcal{N} + \mu - n_M - n_N)\,,$$

where μ *is the (algebraic and geometric) multiplicity of the eigenvalue* $+1$ *of the zero-energy scattering matrix* $S(0)$, *and* n_M *and* n_N *are, respectively, the number of mixed and Neumann boundary conditions.*

We denote by

$$\xi(E; H_{A,B}, H_0)$$

Krein's spectral shift function
In the following theorem we give a Levinson's theorem for the spectral shift function.
Theorem. [2] T. Aktosun, R. W. 2020 *Suppose that V satisfies*

$$\int_0^\infty dx\,(1+x)\,|V(x)| < +\infty,$$

and let $\mathcal{N} < \infty$ be the number of bound states of $H_{A,B}$ including multiplicities. Then, denoting
$\xi(0+; H_{A,B}, H_0) := \lim_{E\downarrow 0} \xi(E; H_{A,B}, H_0)$*,*

$$\xi(0+, H_{A,B}, H_0) = \frac{1}{2}[n-\mu] - \mathcal{N},$$

where μ is the (algebraic and geometric) multiplicity of the eigenvalue $+1$ of the zero-energy scattering matrix $S_{A,B}(0)$.

Buslaev-Faddeev trace formulae

The $b_l(x), l = 0, 1, 2, \cdots$ are C^∞ functions defined by the following recurrent relation,

$$b_0(k,x) = 1, \qquad b_{l+1}(x) = -b_l'(x) - \int_x^\infty V(y)\, b_l(y)\, dy.$$

Moreover,

$$\begin{aligned} c_{-1} &= -\frac{1}{2} A, \\ c_0 &= B - \frac{1}{2} b_1(0)^\dagger A, \\ c_l &= b_l(0)^\dagger B - \left(\frac{1}{2} b_{l+1}(0)^\dagger] + b'(0)_l^\dagger \right) A, l = 1, 2, \cdots . \end{aligned}$$

$$d_l := \sum_{\sigma\in\mathcal{P}} \operatorname{sign}\sigma \sum_{l_i\geq -1: l_1+l_2+\cdots l_n=l} (c_{l_1})_{1,\sigma_1}\,(c_{l_2})_{2,.\sigma_2}\cdots(c_{l_n})_{n,\sigma_n},$$

$$d_j = 0 \text{ if } j < -n_M - n_N, \qquad \text{and } d_{-n_M-n_N} = \frac{c_2}{(2i)^{n_M+n_N}}.$$

$$e_1 = \frac{(2i)^{n_M+n_N}}{c_2}\, d_{1-n_M-n_N},$$

$$e_l = \frac{(2i)^{n_M+n_N}}{c_2}\left(d_{l-n_M-n_N} - \frac{1}{l}\sum_{j=1}^{l-1} j\, d_{l-n_M-n_N-j}\; e_j\right), l = 2,\cdots.$$

$$\ln_e h(k) := \frac{1}{2}\left(\ln h(k) + \ln h(-k)\right), \quad \ln_o h(k) := \frac{1}{2}\left(\ln h(k) - \ln h(-k)\right)$$

Theorem. [2] T. Aktosun, R. W., 2020 *Suppose that*

$$\int_0^\infty (1+x)\, dx\, |V(x)| < +\infty,$$

that V *is infinitely differentiable on* $(0,\infty)$ *and,*

$$\left\| \frac{d^j}{dx^j} V(x) \right\| \leq C_j\, (1+|x|)^{-\rho-j} \textit{for some } \rho \in (1,2], \textit{ and all } j = 0,1,2,\cdots.$$

Then,

$$\widetilde{\sum}_{l=1}^{\mathcal{N}} |\kappa_l| - \frac{1}{\pi} \int_0^\infty \ln_e h(k)\, dk = \frac{e_1}{4},$$

$$\widetilde{\sum}_{l=1}^{\mathcal{N}} \kappa_l^{2j+1} + (-1)^{j+1} \frac{2j+1}{\pi} \int_0^\infty (\ln_e h(k) - \sum_{l=1}^{j} (-1)^l\, e_{2l}\, \frac{1}{(2k)^{2l}})\, k^{2j}$$
$$= \frac{(2j+1)\, e_{2j+1}}{2^{2j+2}}, j = 1,2,\cdots.$$

Furthermore,

$$\widetilde{\sum}_{l=1}^{\mathcal{N}} \kappa_l^{2j} + (-1)^j \frac{2j}{\pi} \int_0^{\infty} (-i \ln_o h(k) - \sum_{l=0}^{j-1} (-1)^{l+1} e_{2l+1} \frac{1}{(2k)^{2l+1}}) k^{2j-1}$$

$$= -j \frac{e_{2j}}{2^{2j}}, j = 1, 2, \cdots .$$

Where, $\widetilde{\sum}_{l=1}^{\mathcal{N}} \kappa_l^q$*, with, respectively,* $q = 1, q = 2j + 1$*, and* $q = 2j$*, means the sum over the absolute value of the eigenvalues,* $-\kappa_l^2$*, to the power* $q/2$*, and repeated according to its multiplicity. Furthermore,* $\mathcal{N} < \infty$ *is the total number of (repeated) eigenvalues.*

Characterization of the scattering data. Marchenko theory

Suppose that

$$\int_0^\infty (1+x)\,|V(x)| < \infty.$$

The scattering data is defined as,

$$\mathcal{M} := \left\{ S(k),\ k \in \mathbf{R}, \{\kappa_j, M_j\}_{j=1}^N \right\},$$

where $S(k)$ is the scattering matrix, and $-\kappa_j^2, j = 1, 2, \cdots, N$, are the eigenvalues, with $\kappa_j > 0$, and the $M_j, j = 1, 2, \cdots, N$, the normalization matrices, are nonnegative matrices so that the matrix,

$$\psi_j(x) := f(i\kappa_j, x)\, M_j$$

has m_j linearly independent columns, where m_j is the multiplicity of the eigenvalue $-\kappa_j^2$. $f(i\kappa_j, x)$ is the Jost solution.

Theorem I. [2] T. Aktosun and R. W. 2020

The set $\left\{ S(k),\ k \in \mathbf{R}, \{\kappa_j, M_j\}_{j=1}^{N} \right\}$, where $S(k), k \in \mathbf{R}$ is a $n \times n$ matrix, $\kappa_j, j = 1, 2, \cdots, N$ are positive distinct numbers and $M_j, j = 1, 2, \cdots, N$ are nonnegative $n \times n$ matrices with rank m_j , are the scattering data of a unique selfadjoint matrix potential, V, satisfying,

$$\int_0^\infty (1 + x)\, |V(x)| < \infty.$$

with eigenvalues $-\kappa_j^2$ and normalization matrices $M_j, j = 1, 2, \cdots, N$ and with boundary matrices (A, B) unique up to multiplication on the right by an invertible matrix, if and only if the following conditions are satisfied.

- (1_s) The scattering matrix $S(k)$ satisfies

$$S(-k) = S(k)^{\dagger} = S(k)^{-1}, \quad k \in \mathbf{R},$$

and there exist constant $n \times n$, matrices S_∞, G_1 such that,

$$S(k) = S_\infty + \frac{G_1}{ik} + o\left(\frac{1}{k}\right), \quad k \to \pm\infty.$$

Moreover,

$$F_s(y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} [S(k) - S_\infty] \, e^{iky} \, dy \in L^2(\mathbf{R}),$$

is bounded on $\mathbf{R}$ and integrable on $\mathbf{R}^+$.

- (2_s) The derivative $F_s'(y)$ exists for a. e. $y \in \mathbf{R}^+$ and it satisfies

$$\int_0^\infty dy\,(1+y)\,\left\|F_s'(y)\right\| < \infty.$$

- (3_s) $F_s'(y), y \in \mathbf{R}^-$ can be written as the sum of two functions, one that is integrable on $\mathbf{R}^-$ and the other that is square integrable on $\mathbf{R}^-$. Furthermore, the only solution in $L^2(\mathbf{R}_-)$ to,

$$-X(y) + \int_{-\infty}^{0} dz\, X(z)\, F_s(z+y) = 0, \quad y \in \mathbf{R}^-,$$

is the trivial solution.

- (4) Denote,

$$F(y) := F_s(y) + \sum_{j=1}^{N} M_j^2 \, e^{-\kappa_j y}.$$

Then, the only solution in $L^1(\mathbf{R}^+)$ to

$$X(y) + \int_0^\infty X(z) \, F(z+y) \, dy = 0, \quad y \in \mathbf{R}^+,$$

is the trivial solution.

- (5) Denote, $\mathcal{N} := \sum_{j=1}^{N} \mathrm{rank}\, M_j$. Then, the number of linearly independent solutions in $L^1(\mathbf{R}^+)$ to

$$X(y) + \int_0^\infty X(z)\, F_s(z+y)\, dy = 0,$$

is equal to $\mathcal{N}$.

Brief idea of the proof

Our precise low- and high-energy estimates allow us to prove that (1_2, 2_s, 3_s, 4) and (5) are necessary conditions.

- Using $(1_s, 2_s)$ and (4) we prove that the Marchenko equation,

$$K(x,y)+F(x,y)+\int_x^\infty dz\, K(x,z)\, F(z+y) = 0, \quad 0 \le x < y,$$

has a unique solution such that for x fixed, $K(x,y) \in L^1(x,\infty)$.

$$V(x) := -2\frac{d}{dx}K(x,x).$$

- $$f(k,x) := e^{ikx} + \int_x^{\infty} K(x,y)\, e^{ikx}\, dx,$$

  the Jost solution

- $$\Psi(k,x) := f(-k,x) + f(k,x)\,S(k),$$
  the physical solution.

- $\Psi_j(x) = f(i\kappa_j, x)\, M_j, j = 1, 2, \cdots, N,$ the bound state eigenfunctions.

  The boundary matrices A, B

  Using the high energy limit of $S(k)$ we obtain the necessary condition

$$\begin{cases} (I - S_\infty)\, A = 0, \\ (I + S_\infty) B = [G_1 - S_\infty K(0,0) - K(0,0) S_\infty] A. \end{cases}$$

These equations have a unique solution (modulo multiplication on the right by an invertible matrix) such that

$$-B^{\dagger}A + A^{\dagger}B = 0,$$

$$A^{\dagger}A + B^{\dagger}B > 0.$$

We prove that if (3_s) and (5) hold, then,

$$-B^{\dagger}\Psi(k,0) + A^{\dagger}\psi'(k,0) = 0, \quad k \in \mathbf{R}.$$

$$-B^{\dagger}\,\Psi_j(0) + A^{\dagger}\,\Psi_j'(0) = 0, \quad j = 1, 2, \cdots, N$$

Theorem II. T. Aktosun and R. W. 2020 The set $\left\{ S(k),\ k \in \mathbf{R}, \{\kappa_j, M_j\}_{j=1}^{N} \right\}$, where $S(k), k \in \mathbf{R}$ is a $n \times n$ matrix, $\kappa_j, j = 1, 2, \cdots, N$ are positive distinct numbers and $M_j, j = 1, 2, \cdots, N$ are nonnegative $n \times n$ matrices with rank m_j, are the scattering data of a unique selfadjoint matrix potential, V, satisfying,

$$\int_0^{\infty} (1 + x)\, |V(x)| < \infty.$$

with eigenvalues $-\kappa_j^2$ and normalization matrices $M_j, j = 1, 2, \cdots, N$ and with boundary matrices (A, B) unique up to multiplication on the right by an invertible matrix, if and only if the following conditions are satisfied.

- $(\tilde{1}_s)$. The scattering matrix $S(k)$ satisfies

$$S(-k) = S(k)^\dagger = S(k)^{-1}, \quad k \in \mathbf{R}.$$

The following limits exist

$$S_\infty := \lim_{k \to \pm\infty} S(k),$$

and

$$S(k) - S_\infty \in L^2(\mathbf{R}).$$

Moreover,

$$F_s(y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} [S(k) - S_\infty] \, e^{iky} \, dy \in L^2(\mathbf{R}),$$

is bounded on $\mathbf{R}$ and integrable on $\mathbf{R}^+$.

- ($\tilde{2}_s$) The derivative $F_s'(y)$ exists for a. e. $y \in \mathbf{R}^+$ and it satisfies
$$\int_0^\infty dy\,(1+y)\,\left\|F_s'(y)\right\| < \infty.$$

- ($\tilde{3}_s$) Denote,

$$L_S^2 := \left\{ \varphi \in L^2(\mathbf{R}) : \varphi(k) = S(-k)\,\varphi(-k), \quad k \in \mathbf{R} \right\}.$$

There is a set $\mathcal{L}$ that is dense in L_S^2 such for any $g \in \mathcal{L}$ there is a h in the Hardy space $\mathbf{H}^2(\mathbf{C}^+)$ that is a solution of the following inhomogeneous Riemann-Hilbert problem.

$$g(k) = h(k) + S(-k)\,h(-k), \quad k \in \mathbb{R}.$$

- ($\tilde{4}$) The only solution $X(y) \in L^1(\mathbf{R}^+)$ to the integral equation

$$X(y) + \int_0^\infty dz\, X(z)\, F(z+y) = 0, \qquad y \in \mathbf{R}^+$$

is the trivial solution $X(y) \equiv 0$. Here, $F(y)$ is the $n \times n$ matrix defined as

$$F(y) := F_s(y) + \sum_{j=1}^{N} M_j^2\, e^{-\kappa_j y}, \quad y \in \mathbf{R}^+.$$

- ($\tilde{5}$) The homogeneous Riemann-Hilbert problem,

$$h(k) + S(-k)\, h(-k) = 0, \qquad k \in \mathbf{R},$$

has precisely $\mathcal{N} := \sum_{j=1}^{N} \operatorname{rank} M_j$ linearly independent solutions in the Hardy space $\mathbf{H}^2(\mathbf{C}^+)$.

In Theorem II we reconstruct the boundary matrices (A, B) using the unitarity of the generalized Fourier maps.

$$\mathcal{H} := \mathcal{R} \oplus L^2(0,\infty),$$

$$F\phi := \oplus_{j=1}^{N} \int_0^\infty \Psi_j^\dagger(x)\,\phi(x)\,dx \oplus \frac{1}{\sqrt{2\pi}} \int_0^\infty \psi(k,x)^\dagger\,\phi(x)\,dx.$$

We prove that F is unitary from $L^2(\mathbf{R}^+)$ onto $\mathcal{H}$.

We define the selfadjoint operator, $\hat{H}$, in $\mathcal{H}$,

$$\hat{H}\left(\oplus_{j=1}^{N} v_j \oplus \hat{f}\right) := \oplus_{j=1}^{N}(-\kappa_j^2) v_j \oplus k^2 \hat{f}(k),$$

$$D(\hat{H}) := \mathcal{R} \oplus \left\{ \hat{f} \in L^2(\mathbf{R}^+) : k^2\, \hat{f}(k) \in L^2(\mathbf{R}^+) \right\}.$$

We denote by H the operator,

$$H := F^{\dagger}\, \hat{H}\, F, \quad D(H) := \left\{ \phi \in L^2(\mathbf{R}^+) : F\, \phi \in D(\hat{H}) \right\}.$$

We prove that H is a selfadjoint extension of $-\Delta + V$. Then, there exist (A, B) such that,

$$-B^{\dagger}A + A^{\dagger}B = 0,$$

$$A^{\dagger}A + B^{\dagger}B > 0,$$

so that,

$$-B^{\dagger}\psi(0) + A^{\dagger}\psi'(0) = 0, \quad \forall \psi \in D(H).$$

Using this fact we prove that the physical solution $\Psi(k, x)$ and the bound state eigenfunctions, $\Psi_j(x), j = 1, 2, \cdots, N$, satisfy the boundary condition.

For a (different) characterization in the particular case of the Dirichlet boundary condition, $\psi(0) = 0$ see

[1] Z. S. Agranovich and V. A. Marchenko 1963.

Let U we the following matrix

$$U := (B - iA)\,(B + iA)^{-1}.$$

$B + iA$ is invertible and U is unitary. Let M be a unitary matrix that diagonalizes U,

$$M^{\dagger}\,U\,M = \mathrm{diag}\left\{ e^{2\cos\theta_1}, e^{2\cos\theta_2}, \cdots e^{2\cos\theta_n} \right\},$$

where $0 < \cos\theta_j \leq \pi$. In general there are n_{N} values with $\theta_j = \pi/2$ and n_{D} values with $\theta_j = \pi$, and hence there are n_{M} remaining values, with $n_{\mathrm{M}} := n - n_{\mathrm{N}} - n_{\mathrm{D}}$, such that the corresponding θ_j-values lie in $(0, \pi/2) \cup (\pi/2, \pi)$.

We denote by $n_{M,\mathrm{b}}$ the number of θ_j with $0 < \theta_j < \pi/2$.
We designate by $\tan\Theta$ the following diagonal $n \times n$ matrix,

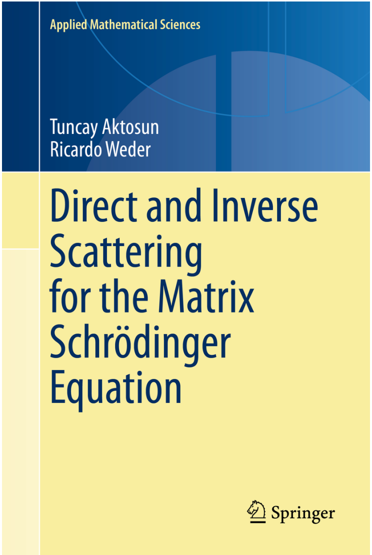
$$\tan\Theta := \mathrm{diag}\left\{\widehat{\tan\theta_1}, \widehat{\tan\theta_2}, \cdots, \widehat{\tan\theta_n}\right\},$$

where $\widehat{\tan\theta_j} = 0$ if $0 < \theta_j \le \pi/2$, $\widehat{\tan\theta_j} = \tan\theta_j$ if $\pi/2 < \theta_j < \pi$, and $\widehat{\tan\theta_j} = 0$ if $\theta_j = \pi$, for $j = 1, 2, \cdots, n$.

Theorem. Bargmann-Birman-Schwinger estimate. [2] T. Aktosun and R. W. 2020

Let us denote by $N_{A,B}$ the number of negative eigenvalues of $H_{A,B}$. Then,

$$N_{A,B} \leq n_{M,\mathrm{b}} + n_N + \int_0^\infty \mathrm{trace}\left[V(x)\,(x - \tan\Theta)\right]\, dx.$$


Applied Mathematical Sciences
Tuncay Aktosun
Ricardo Weder
Direct and Inverse Scattering for the Matrix Schrödinger Equation
Springer

THANKS FOR YOUR ATTENTION

References

[1] Z. S. Agranovich and V. A. Marchenko, *The Inverse Problem of Scattering Theory, Gordon and Breach, New York, 1963.*
[2] T. Aktosun and R. Weder, *Direct and Inverse Scattering for the Matrix Schrödinger Equation,* Applied Mathematical Sciences **203** , Springer Verlag New York, 2021 (published in May 2020).
[3] K. Ammari, and M. Sabri, Dispersion for Schrödinger operators on regular trees, arXiv: 2009.03153v1 [math.AP] (2020)
[4] D. Bambusi, and Z. Zhao, Dispersive estimates for quasi-periodic Schrödinger operators on $1-d$ lattices, Advances in Mathematics **366**, (2020), 107071.

References

[5] J. Bourgain, *Global Solutions of Nonlinear Schrödinger Equations,* Colloquium Publications **46**, A.M.S., Providence R. I., 1999.
[6] P. D'Ancona and S. Selberg, Dispersive estimates for the 1D Schrödinger equation with a steplike potential, J. Differential Equations **252**, 1603–1634 (2012)
[7] I. E. Egorova, E. A. Kopylova, V. A. Marchenko and G. Teschl, Dispersion estimates for one-dimensional Schrödinger and Klein-Gordon equations revisited, Russian Math. Surveys **71**, 391–415 (2016).
8] I. Egorova, E.A. Kopylova, and G. Teschl, Dispersion estimates for one-dimensional discrete Schrödinger and wave equations, J. Spectral Theory **5** (2015), 663Ð696.

References

[9] L. Fanelli, Dispersive equations in quantum mechanics, Rend. Mat. Appl. (7) **28**, 237–384 (2008).
[10] M. Goldberg and W. Schlag, Dispersive estimates for Schrödinger operators in dimensions one and three, Commun. Math. Phys. **251**, 157–178 (2004).
[11] A. Grecu, and L. I. Ignat, The Schrödinger equation on a star-shaped graph under general coupling conditions, J. Phys. A: Math. Theor. **52**, 035202 (2019).
[12] M. S. Harmer, Inverse scattering for the matrix Schrödinger operator and Schrödinger operator on graphs with general self-adjoint boundary conditions, ANZIAM J. **44**, 161–168 (2002).

References

[13] M. S. Harmer, *The Matrix Schrödinger Operator and Schrödinger Operator on Graphs,* Ph.D. thesis, University of Auckland, New Zealand, 2004.

[14] M. Holzleitner, A. Kostenko and G. Teschl, Dispersion estimates for spherical Schrödinger equations: the effect of the boundary conditions, Opuscula Math. **36**, 769–786 (2016).

[15] M. Holzleitner, A. Kostenko and G. Teschl, Dispersion estimates for spherical Schrödinger equations with critical angular momentum. In *Partial Differential Equations, Mathematical Physics, and Stochastic Analysis: The Helge Holden Anniversary Volume.* Editors F. Gesztesy, H. Hanche-Olsen, E. R. Jakobsen, Y. Lyubarskii, N. H. Risebro, and K. Seip, pp 319-347, EMS Congress Reports **14**, 2018.

[16] J. L. Journé, A. Soffer, and C. D. Sogge, Decay estimates for Schrödinger operators, Commun. Pure Appl. Math. **44** (1991), 573-604.

References

[17] E. A. Kopylova and G. Teschl, Dispersion estimates for one-dimensional discrete Dirac equations, J. Math. Anal. Appl. **434**,191–208 (2016).
[18] E. A. Kopylova, and G. Teschl, Scattering properties and dispersion estimates for a one-dimensional discrete Dirac equation, arXiv: 2001.08445v1 [math.SP] (2020
[19] A. Kostenko, G. Teschl and J. Toloza, Dispersion estimates for spherical Schrödinger equations, Ann. Henri Poincaré **17**, 3147–3176 (2016).
[20] V. Kostrykin and R. Schrader, Kirchhoff's rule for quantum wires, J. Phys. A **32** (1999) , 595–630

References

[21] V. Kostrykin and R. Schrader, Kirchhoff's rule for quantum wires. II: The inverse problem with possible applications to quantum computers, Fortschr. Phys. **48**, (2000), 703–716 (2000).
[22] M. Maeda, H. Sasaki, E. Segawa, A. Suzuki, and K. Suzuki, Scattering and inverse scattering for nonlinear quantum walks, Discrete Contin. Dyn. Syst.**38** (2018), 3687Ð3703.
[23] M. Maeda, H. Sasaki, E. Segawa, A. Suzuki, and K, Suzuki, Dispersive estimates for quantum walks on 1D lattice, arXiv: 1808.05714v2 [math-ph] (2020).
[24] F. Ali Mehmeti, K. Ammari and S. Nicaise, Dispersive effects and high frequency behaviour for the Schrödinger equation in star-shaped networks, Port. Math. **72**, 309-355 (2015).

References

[25] I. Naumkin, R. Weder, $L^p - L^{p'}$ estimates for matrix Schrödinger equations, Journal of Evolution Equations, online first https://doi.org/10.1007/s00028-020-00605-x, (2020).
[26] O. Prill, Dispersive estimates for solutions to the perturbed one-dimensional Klein-Gordon equation with and without one-gap periodic potential, Math. Nachr. **287**, 1456–1496 (2014).
[27] F. S. Rofe-Beketov, and A. M. Kholkin, Spectral Analysis of Differential Operators. Interplay Between Spectral and Oscillatory Properties. World Scientific, Singapore, 2005.
[28] W. Schlag, Dispersive estimates for Schrödinger operators: a survey, Mathematical Aspects of Nonlinear Dispersive Equations, 255-285, Ann. Math. Stud. **163**, Princeton Univ. Pres, NJ, 2007.

[29] A. Stefanov, and P. G. Kevrekidis, Asymptotic behaviour of small solutions for the discrete nonlinear Schrödinger and Klein-Gordon equations, Nonlinearity **18**, 2007, 1841-1857.
[30] R. Weder, The W_{kp}-continuity of the Schrödinger wave operators on the line, Comm. Math. Phys. **208** (1999), 507–520.
[31] R. Weder, $L^p - L^{p'}$ estimates for the Schrödinger equation on the line and inverse scattering for the non linear Schrödinger equation a potential, J. Funct. Analysis **170** (2000), 37–68.
[32] R. Weder, Inverse Scattering on the line for the nonlinear Klein-Gordon equation with a potential, J. Math. Anal. Applications **252** (2000), 102-123.

References

[33] R. Weder, Center manifold for nonintegrable nonlinear Schrödinger equations on the line, Comm. Math. Phys. **215** (2000), 343-356.
[34] R. Weder, The $L^p - L^{\acute{p}}$ estimate for the Schrödinger equation on the half-line, Journal of Mathematical Analysis and Applications **281** (2003), 233-243.
[35] R. Weder, The L^p boundedness of the wave operators for matrix Schrödinger equations, arXiv:1912.12793v2 [math-ph] (2019),
[36] K. Yajima, The $W_{k,p}-$continuity of wave operators for SchrÖdinger operators. J. Math. Soc. Japan **47** (1995), 551-581.
[37] K. Yajima, The L_p boundedness of the wave operators for two dimensional Schrödinger operators with threshold singularities, arXiv:2008.07906 [math.AP] (2020).