Higher dimensional summability and Lebesgue points

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Wiener amalgam spaces

The space $L_p(\mathbb{R}^2)$ is equipped with the norm

$$\|f\|_p := \begin{cases} \left(\int_{\mathbb{R}^2} |f|^p \, d\lambda\right)^{1/p}, & 0 < p < \infty; \\ \sup_{\mathbb{R}^2} |f|, & p = \infty. \end{cases}$$

A measurable function f belongs to the *Wiener amalgam space* $W(L_p, \ell_q)(\mathbb{R}^2)$ $(1 \leq p, q \leq \infty)$ if

$$\|f\|_{W(L_p,\ell_q)} := \left(\sum_{k \in \mathbb{Z}^2} \|f(\cdot + k)\|_{L_p[0,1)^2}^q \right)^{1/q} < \infty,$$

with the obvious modification for $q = \infty$.
It is easy to see that $W(L_p, \ell_p)(\mathbb{R}^2) = L_p(\mathbb{R}^2)$ and

$$W(L_\infty, \ell_1)(\mathbb{R}^2) \subset L_p(\mathbb{R}^2) \subset W(L_1, \ell_\infty)(\mathbb{R}^2) \qquad (1 \leq p \leq \infty).$$

Dirichlet integrals

The *Fourier transform* of $f \in L_1(\mathbb{R})$ is given by

$$\widehat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(s) e^{-\imath xs} \, ds \qquad (x \in \mathbb{R}),$$

where $\imath = \sqrt{-1}$. If $f \in L_p(\mathbb{R})$ for some $1 \leq p \leq 2$, then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f}(s) e^{\imath xs} \, ds \qquad (x \in \mathbb{R})$$

holds if $\widehat{f} \in L_1(\mathbb{R})$. This motivates the definition

$$s_T f(x) := \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \widehat{f}(s) e^{\imath xs} \, ds.$$

Theorem (Carleson, Hunt, 1967)

For $f \in L_p(\mathbb{R})$, $1 < p < \infty$,

$$\lim_{T \to \infty} s_T f = f \qquad a.e.$$

One-dimensional summability

This convergence does not hold for $p = 1$. Let

$$\sigma_T^\theta f(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \theta\left(\frac{|s|}{T}\right) \widehat{f}(s) e^{\imath xs}\, ds.$$

For $\theta(s) = \max((1-|s|), 0)$ we obtain the Fejér means:

$$\begin{aligned} \sigma_T f(x) &:= \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \left(1 - \frac{|s|}{T}\right) \widehat{f}(s) e^{\imath xs}\, ds \\ &= \frac{1}{T} \int_0^T s_t f(x)\, dt. \end{aligned}$$

The Hardy-Littlewood maximal function is defined by

$$Mf(x) := \sup_{h>0} \frac{1}{2h} \int_{-h}^{h} |f(x-s)| \, ds.$$

Theorem

We have

$$\sup_{\rho>0} \rho \lambda(Mf > \rho) \leq C \|f\|_1 \qquad (f \in L_1(\mathbb{R})),$$

and

$$\|Mf\|_p \leq C_p \|f\|_p \qquad (f \in L_p(\mathbb{R}), 1 < p \leq \infty).$$

Corollary

If $f \in L_1(\mathbb{R})$, *then*

$$\lim_{h \to 0} \frac{1}{2h} \int_{-h}^{h} f(x-s)\, ds = f(x) \qquad a.e.\ x \in \mathbb{R}.$$

Then

$$\lim_{h \to 0} \frac{1}{2h} \int_{-h}^{h} f(x-s) - f(x)\, ds = 0 \qquad \text{a.e.}$$

and

$$\lim_{h \to 0} \frac{1}{2h} \left| \int_{-h}^{h} f(x-s) - f(x)\, ds \right| = 0 \qquad \text{a.e.}$$

A point $x \in \mathbb{R}$ is called a *Lebesgue point* of $f \in L_1(\mathbb{R})$ if

$$\lim_{h \to 0} \frac{1}{2h} \int_{-h}^{h} |f(x-s) - f(x)|\, ds = 0.$$

Theorem

Almost every point $x \in \mathbb{R}$ is a Lebesgue point of $f \in L_1(\mathbb{R})$.

Theorem (Lebesgue, Fejér, 1905)

For all Lebesgue points of $f \in L_1(\mathbb{R})$

$$\lim_{T\to\infty} \sigma_T f(x) = f(x).$$

Two-dimensional summability

In the two-dimensional case,

$$\widehat{f}(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(s,t) e^{-\imath(xs+yt)}\, dsdt \qquad (x,y \in \mathbb{R})$$

and for $f \in L_p(\mathbb{R}^2)$ $(1 \le p \le 2)$,

$$f(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \widehat{f}(s,t) e^{\imath(xs+yt)}\, ds\, dt \qquad (x,y \in \mathbb{R})$$

if $\widehat{f} \in L_1(\mathbb{R}^2)$.

Let

$$\sigma_T^{q,\theta} f(x,y) := \frac{1}{2\pi} \int_{\mathbb{R}^2} \theta\left(\frac{\|(s,t)\|_q}{T}\right) \widehat{f}(s,t) e^{\imath(xs+yt)}\, ds\, dt \qquad (T>0)$$

with $q = 1, 2, \infty$,

$$\sigma_T^{\theta} f(x,y) := \frac{1}{2\pi} \int_{\mathbb{R}^2} \theta_1\left(\frac{|s|}{T_1}\right) \theta_2\left(\frac{|t|}{T_2}\right) \widehat{f}(s,t) e^{\imath(xs+yt)}\, ds\, dt \quad (T \in \mathbb{R}_+^2).$$

Suppose that $\theta(0) = \theta_i(0) = 1$. This summation contains all well known summability methods, such as the Fejér, Riesz, Weierstrass, Abel, Picard, Bessel summations.

Circular summability

Here

$$\begin{aligned}\sigma_T^{2,\theta} f(x,y) &:= \frac{1}{2\pi}\int_{\mathbb{R}^2} \theta\left(\frac{\|(s,t)\|_2}{T}\right) \widehat{f}(s,t) e^{\imath(xs+yt)}\, ds\, dt \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \theta_0\left(\frac{s}{T},\frac{t}{T}\right) \widehat{f}(s,t) e^{\imath(xs+yt)}\, ds\, dt \qquad (T>0),\end{aligned}$$

where

$$\theta_0(s,t) := \theta(\|(s,t)\|_2)$$

and suppose that $\theta_0 \in L_1(\mathbb{R}^2)$ and $\widehat{\theta_0} \in L_1(\mathbb{R}^2)$.
We denote by $B(c,h)$ ($c \in \mathbb{R}^d, h>0$) the ball $\{x \in \mathbb{R}^d : \|x-c\|_2 < h\}$.
Let the dyadic coronas be defined by

$$Q_k := B(0,2^k) \setminus B(0,2^{k-1}) \quad (k>0), \qquad Q_0 := B(0,1).$$

The Herz space $E_q(\mathbb{R}^d)$ contains all functions f for which

$$\|f\|_{E_q} := \sum_{k=0}^{\infty} 2^{dk(1-1/q)} \|f \mathbf{1}_{Q_k}\|_q < \infty.$$

Then

$$L_1(\mathbb{R}^d) = E_1(\mathbb{R}^d) \supset E_q(\mathbb{R}^d) \supset E_{q'}(\mathbb{R}^d) \supset E_\infty(\mathbb{R}^d), \qquad 1 < q < q' < \infty.$$

The Hardy-Littlewood maximal function is defined by

$$M_p f(x,y) := \sup_{h>0} \left(\frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} |f(x-s,y-t)|^p \, dsdt \right)^{1/p}.$$

Theorem

If $1 \le p < \infty$, then

$$\sup_{\rho>0} \rho \lambda(M_p f > \rho)^{1/p} \le C \|f\|_p \qquad (f \in L_p(\mathbb{R}^2)),$$

$$\|M_p f\|_r \le C_r \|f\|_r \qquad (f \in L_r(\mathbb{R}^2), p < r \le \infty)$$

and

$$\sup_{k\in\mathbb{Z}^2} \sup_{\rho>0} \rho \lambda(M_p f > \rho, [k,k+1))^{1/p} = \|M_p f\|_{W(L_{p,\infty},\ell_\infty)} \le C_p \|f\|_{W(L_p,\ell_\infty)}$$

for all $f \in W(L_p,\ell_\infty)(\mathbb{R}^2)$. Moreover, for every $p < r \le \infty$,

$$\|M_p f\|_{W(L_r,\ell_\infty)} \le C_r \|f\|_{W(L_r,\ell_\infty)} \qquad (f \in W(L_r,\ell_\infty)(\mathbb{R}^2)).$$

Corollary

If $f \in W(L_1, \ell_\infty)(\mathbb{R}^2)$, *then*

$$\lim_{h\to 0} \frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} f(x-s, y-t)\, ds\, dt = f(x,y) \qquad a.e.\ (x,y) \in \mathbb{R}^2.$$

Then

$$\lim_{h\to 0} \frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} f(x-s, y-t) - f(x,y)\, ds\, dt = 0.$$

A point $(x,y) \in \mathbb{R}^2$ is called a *p-Lebesgue point* of f $(1 \le p < \infty)$ if

$$\lim_{h\to 0} \left(\frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} |f(x-s, y-t) - f(x,y)|^p\, ds\, dt \right)^{1/p} = 0.$$

All r-Lebesgue points are p-Lebesgue points, whenever $p < r$.

Theorem

Almost every point $(x,y) \in \mathbb{R}^2$ is a p-Lebesgue point of $f \in W(L_p, \ell_\infty)(\mathbb{R}^2)$ if $1 \leq p < \infty$.

Theorem (Stein, Weiss, 1971, Feichtinger, Weisz, 2006)

Let $\theta_0 \in L_1(\mathbb{R}^2)$, $1 \leq p < \infty$ and $1/p + 1/q = 1$. If $\widehat{\theta}_0 \in E_q(\mathbb{R}^2)$ then for all p-Lebesgue points of $f \in W(L_p, \ell_\infty)(\mathbb{R}^2)$,

$$\lim_{T\to\infty} \sigma_T^{2,\theta} f(x,y) = f(x,y).$$

Theorem (Feichtinger, Weisz, 2006)

Suppose that $\theta_0 \in L_1(\mathbb{R}^2)$, $\widehat{\theta}_0 \in L_1(\mathbb{R}^2)$, $1 \leq p < \infty$ and $1/p + 1/q = 1$. If

$$\lim_{T\to\infty} \sigma_T^{2,\theta} f(x,y) = f(x,y)$$

for all p-Lebesgue points of $f \in L_p(\mathbb{R}^2)$, then $\widehat{\theta}_0 \in E_q(\mathbb{R}^2)$.

Unrestricted rectangular summability

Suppose that $\theta_i \in L_1(\mathbb{R}^2)$, $\widehat{\theta}_i \in L_1(\mathbb{R}^2)$ $(i = 1, 2)$ and $T = (T_1, T_2)$. Then

$$\sigma_T^\theta f(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \theta_1 \left(\frac{|s|}{T_1} \right) \theta_2 \left(\frac{|t|}{T_2} \right) \widehat{f}(s,t) e^{\imath(xs+yt)} \, ds \, dt.$$

Recall that

$$\|f\|_{W(L_p,\ell_\infty)} := \left(\sup_{n,m\in\mathbb{Z}} \int_n^{n+1} \int_m^{m+1} |f(x,y)|^p \, dydx \right)^{1/p}.$$

The norms of the iterated Wiener amalgam spaces $W_I(L_p, \ell_\infty)(\mathbb{R}^2)$ and $W_I(L_p \log L, \ell_\infty)(\mathbb{R}^2)$ $(1 \le p \le \infty)$ are given by

$$\|f\|_{W_I(L_p,\ell_\infty)} := \left(\sup_{n\in\mathbb{Z}} \int_n^{n+1} \sup_{m\in\mathbb{Z}} \int_m^{m+1} |f(x,y)|^p \, dydx \right)^{1/p}$$

and

$$\|f\|_{W_I(L_p \log L,\ell_\infty)} := \left(\sup_{n\in\mathbb{Z}} \int_n^{n+1} \sup_{m\in\mathbb{Z}} \int_m^{m+1} |f(x,y)|^p \log^+ |f(x,y)| \, dydx \right)^{1/p}.$$

A function f is in the set $L_p \log L(\mathbb{R}^2)$ $(1 \leq p < \infty)$ if

$$\|f\|_{L_p \log L} := \left(\int_{\mathbb{R}^2} |f|^p \log^+ |f| \, d\lambda \right)^{1/p} < \infty.$$

Then

$$W(L_p, \ell_\infty)(\mathbb{R}^2) \supset W_l(L_p \log L, \ell_\infty)(\mathbb{R}^2) \supset L_p \log L(\mathbb{R}^2), C_0(\mathbb{R}^2), L_r(\mathbb{R}^2)$$

for all $1 \leq p < r \leq \infty$.

Recall that

$$M_p f(x,y) := \sup_{h>0} \left(\frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} |f(x-s,y-t)|^p \, dsdt \right)^{1/p}$$

and that M_p is of weak type (p,p) and bounded on $L_r(\mathbb{R}^2)$ $(p < r \leq \infty)$. The strong Hardy-Littlewood maximal function is defined by

$$M_{s,p} f(x,y) := \sup_{h_1,h_2>0} \left(\frac{1}{4h_1 h_2} \int_{-h_1}^{h_1} \int_{-h_2}^{h_2} |f(x-s,y-t)|^p \, dsdt \right)^{1/p}.$$

Theorem (Weisz 2014)

If $1 \le p < \infty$ *and* $f \in L_p \log L(\mathbb{R}^2)$, *then*

$$\sup_{\rho>0} \rho\lambda(x : M_{s,p}f(x) > \rho, x \in I)^{1/p} \le C_p + C_p \|f\|_{L_p \log L}$$

$$\|M_{s,p}f\|_r \le C_r \|f\|_r \qquad (f \in L_r(\mathbb{R}^2), p < r \le \infty).$$

If $f \in W_I(L_p \log L, \ell_\infty)(\mathbb{R}^2)$, *then*

$$\|M_{s,p}f\|_{W(L_{p,\infty},\ell_\infty)} \le C_p + C_p \|f\|_{W_I(L_p \log L,\ell_\infty)}$$

and, for $p < r \le \infty$,

$$\|M_{s,p}f\|_{W(L_r,\ell_\infty)} \le C_r \|f\|_{W_I(L_r,\ell_\infty)} \qquad (f \in W_I(L_r, \ell_\infty)(\mathbb{R}^2)).$$

Corollary

If $f \in W_I(L_1 \log L, \ell_\infty)(\mathbb{R}^2)$, *then*

$$\lim_{h\to 0} \frac{1}{4h_1 h_2} \int_{-h_1}^{h_1} \int_{-h_2}^{h_2} f(x-s, y-t)\, ds\, dt = f(x,y) \qquad a.e.\ (x,y) \in \mathbb{R}^2.$$

A point $(x,y) \in \mathbb{R}^2$ is called a strong *p-Lebesgue point* of f $(1 \le p < \infty)$ if

$$\lim_{h\to 0} \left(\frac{1}{4h_1 h_2} \int_{-h_1}^{h_1} \int_{-h_2}^{h_2} |f(x-s, y-t) - f(x,y)|^p \, ds\, dt \right)^{1/p} = 0.$$

Here $h \to 0$ means that $h_1 \to 0$ and $h_2 \to 0$.
Recall that $(x,y) \in \mathbb{R}^2$ is a *p-Lebesgue point* of f $(1 \le p < \infty)$ if

$$\lim_{h\to 0} \left(\frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} |f(x-s, y-t) - f(x,y)|^p \, ds\, dt \right)^{1/p} = 0.$$

Theorem

Almost every point $(x, y) \in \mathbb{R}^2$ *is a strong p-Lebesgue point of* $f \in W_I(L_p \log L, \ell_\infty)(\mathbb{R}^2)$ *if* $1 \le p < \infty$.

Theorem (Weisz, 2014)

Let $\theta_i \in L_1(\mathbb{R})$, $1 \le p < \infty$ *and* $1/p + 1/q = 1$. *If* $\widehat{\theta}_i \in E_q(\mathbb{R})$ $(i = 1, 2)$, *then*

$$\lim_{T \to \infty} \sigma_T^\theta f(x, y) = f(x, y)$$

for all strong p-Lebesgue points of $f \in W_I(L_p \log L, \ell_\infty)(\mathbb{R}^2)$.

Here $T \to \infty$ means that $T_1 \to \infty$ and $T_2 \to \infty$. Note that

$$W_I(L_p \log L, \ell_\infty)(\mathbb{R}^2) \supset L_p \log L(\mathbb{R}^2), C_0(\mathbb{R}^2), L_r(\mathbb{R}^2)$$

for all $1 \le p < r \le \infty$.

Restricted rectangular summability

$$\sigma_T^\theta f(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \theta_1\left(\frac{|s|}{T_1}\right)\theta_2\left(\frac{|t|}{T_2}\right)\widehat{f}(s,t)e^{\imath(xs+yt)}\,ds\,dt.$$

Suppose that $\tau \geq 0$ and

$$T = (T_1, T_2) \in \mathbb{R}^2_\tau := \{x \in \mathbb{R}^2_+ : 2^{-\tau} \leq x_1/x_2 \leq 2^\tau\}.$$

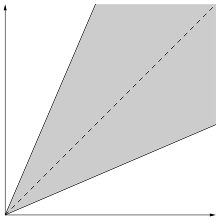

Figure: The cone for $d = 2$.

The weighted Herz space $E_q^\omega(\mathbb{R})$ contains all functions f for which

$$\|f\|_{E_q^\omega} := \sum_{k=0}^{\infty} 2^{k(\omega+1-1/q)} \|f \mathbf{1}_{Q_k}\|_q < \infty,$$

where

$$Q_k := B(0, 2^k) \setminus B(0, 2^{k-1}) \quad (k > 0), \qquad Q_0 := B(0, 1).$$

Then

$$E_q(\mathbb{R}) = E_q^0(\mathbb{R}) \supset E_q^\omega(\mathbb{R}) \qquad 0 < \omega < \infty$$

and

$$L_1(\mathbb{R}) \supset E_1^\omega(\mathbb{R}) \supset E_q^\omega(\mathbb{R}) \supset E_{q'}^\omega(\mathbb{R}) \supset E_\infty^\omega(\mathbb{R}), \qquad 1 < q < q' < \infty.$$

Recall that

$$M_{s,p}f(x,y) := \sup_{h_1,h_2>0} \left(\frac{1}{4h_1h_2} \int_{-h_1}^{h_1} \int_{-h_2}^{h_2} |f(x-s,y-t)|^p \, dsdt \right)^{1/p}.$$

Define

$$\begin{aligned} \mathcal{M}_p^{\omega} f \quad &:= \quad \sup_{i,j\in\mathbb{N},h>0} 2^{-\omega(i+j)} \\ &\qquad \left(\frac{1}{4\cdot 2^{i+j}h^2} \int_{-2^i h}^{2^i h} \int_{-2^j h}^{2^j h} |f(x-s,y-t)|^p \, ds\, dt \right)^{1/p}. \end{aligned}$$

Same results as for M_p.

Theorem (Weisz 2016)

If $1 \leq p < \infty$ *and* $\omega > 0$, *then*

$$\sup_{\rho>0} \rho\lambda(\mathcal{M}_p^\omega f > \rho)^{1/p} \leq C\|f\|_p \qquad (f \in L_p(\mathbb{R}^2)),$$

$$\left\|\mathcal{M}_p^\omega f\right\|_r \leq C_r \left\|f\right\|_r \qquad (f \in L_r(\mathbb{R}^2), p < r \leq \infty)$$

and

$$\sup_{k\in\mathbb{Z}^2} \sup_{\rho>0} \rho\lambda(\mathcal{M}_p f > \rho, [k, k+1))^{1/p} = \left\|\mathcal{M}_p^\omega f\right\|_{W(L_{p,\infty},\ell_\infty)} \leq C_p \left\|f\right\|_{W(L_p,\ell_\infty)}$$

for all $f \in W(L_p, \ell_\infty)(\mathbb{R}^2)$. *Moreover, for every* $p < r \leq \infty$,

$$\left\|\mathcal{M}_p^\omega f\right\|_{W(L_r,\ell_\infty)} \leq C_r \left\|f\right\|_{W(L_r,\ell_\infty)} \qquad (f \in W(L_r, \ell_\infty)(\mathbb{R}^2)).$$

Recall the definition of the Lebesgue and strong Lebesgue points:

$$\lim_{h\to 0}\left(\frac{1}{4h^2}\int_{-h}^{h}\int_{-h}^{h}|f(x-s,y-t)-f(x,y)|^p\,ds\,dt\right)^{1/p}=0,$$

$$\lim_{h\to 0}\left(\frac{1}{4h_1h_2}\int_{-h_1}^{h_1}\int_{-h_2}^{h_2}|f(x-s,y-t)-f(x,y)|^p\,ds\,dt\right)^{1/p}=0.$$

The definition of the strong Lebesgue points is equivalent to

$$\lim_{r\to 0} \sup_{h_1<r,h_2<r} \left(\frac{1}{4h_1h_2} \int_{-h_1}^{h_1} \int_{-h_2}^{h_2} |f(x-s,y-t) - f(x,y)|^p \, ds \, dt \right)^{1/p} = 0.$$

A point $(x,y) \in \mathbb{R}^2$ is a (p,ω)-*Lebesgue point* of f if

$$\lim_{r\to 0} \sup_{\substack{i,j\in\mathbb{N},h>0 \\ 2^i h<r, 2^j h<r}} 2^{-\omega(i+j)} \left(\frac{1}{4\cdot 2^{i+j}h^2} \int_{-2^i h}^{2^i h} \int_{-2^j h}^{2^j h} |f(x-s,y-t) - f(x,y)|^p \, ds \, dt \right)^{1/p} = 0.$$

Theorem (Weisz 2016)

Almost every point $(x,y)\in\mathbb{R}^2$ is a (p,ω)-Lebesgue point of $f\in W(L_p,\ell_\infty)(\mathbb{R}^2)$ if $1\le p<\infty$ and $\omega>0$.

Theorem (Weisz, 2016)

Let $\theta_i\in L_1(\mathbb{R})$, $1\le p<\infty$ and $1/p+1/q=1$ and $\omega>0$. If $\widehat{\theta}_i\in E_q^\omega(\mathbb{R})$ $(i=1,2)$, then

$$\lim_{T\to\infty,T\in\mathbb{R}^2_\tau}\sigma_T^\theta f(x,y)=f(x,y)$$

for all (p,ω)-Lebesgue points of $f\in W(L_p,\ell_\infty)(\mathbb{R}^2)$.

The a.e. convergence for $L_1(\mathbb{T})$ and for Fejér means was proved by Marcinkiewicz and Zygmund (1939).

Cubic summability

Suppose that θ is continuous on $\mathbb{R}_+$, the support of θ is $[0, c]$ for some $0 < c \leq \infty$ and θ is differentiable on $(0, c)$. Suppose further that

$$\int_0^\infty (t \vee 1)^2 |\theta'(t)| \, dt < \infty, \qquad \lim_{t\to\infty} t^2\theta(t) = 0,$$

where $\vee$ denotes the maximum and $\wedge$ the minimum. Assume also that

$$\left|\int_0^\infty \theta'(t) \cos(ts) \, dt\right| \leq Cs^{-\alpha}, \qquad \left|\int_0^\infty \theta'(t) \, t \sin(ts) \, dt\right| \leq Cs^{-\alpha}.$$

The cubic or Marcinkiewicz-θ-means are defined by

$$\sigma_T^{\infty,\theta} f(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \theta\left(\frac{|s| \vee |t|}{T}\right) \widehat{f}(s, t) e^{\imath(xs+yt)} \, ds \, dt.$$

Recall that $(x, y) \in \mathbb{R}^2$ is a (p, ω)-*Lebesgue point* if for all $\omega > 0$

$$\lim_{r\to 0} \sup_{\substack{i,j\in\mathbb{N},h>0\\ 2^i h<r,2^j h<r}} 2^{-\omega(i+j)} \left(\frac{1}{4\cdot 2^{i+j}h^2}\int_{-2^i h}^{2^i h}\int_{-2^j h}^{2^j h} |f(x-s,y-t)-f(x,y)|^p \, ds\, dt\right)^{1/p} = 0.$$

If in addition

$$\lim_{r\to 0} \sup_{\substack{i,j\in\mathbb{N},h>0\\ 2^i h<r,2^j h<r}} 2^{-\omega(i+j)} \left(\frac{1}{4\cdot 2^{i+j}h^2}\int_{-2^i h}^{2^i h}\int_{s-2^j h}^{s+2^j h} |f(x-s,y-t)-f(x,y)|^p \, dt\, ds\right)^{1/p} = 0,$$

then we say that $(x, y) \in \mathbb{R}^2$ is a *strong* (p, ω)-*Lebesgue point*.

Theorem (Weisz, 2015)

Almost every point $(x,y) \in \mathbb{R}^2$ *is a strong* (p,ω)*-Lebesgue point of* $f \in W(L_p,\ell_\infty)(\mathbb{R}^2)$ *if* $1 \le p < \infty$ *and* $\omega > 0$.

Theorem (Weisz, 2015)

Suppose that $1 < p < \infty$, $1/p + 1/q = 1$ *and* $0 < \omega < \min(\alpha/2, 1/(2q))$. *If* $f \in W(L_p,\ell_\infty)(\mathbb{R}^2)$, *then*

$$\lim_{T\to\infty} \sigma_T^{\infty,\theta} f(x,y) = f(x,y)$$

for all (p,ω)*-Lebesgue points of* f.

Theorem (Weisz, 2015)

If $f \in W(L_1, \ell_\infty)(\mathbb{R}^2)$ *and* $0 < \omega < \min(\alpha, 1)/2$, *then*

$$\lim_{T\to\infty} \sigma_T^{\infty,\theta} f(x,y) = f(x,y)$$

for all strong $(1, \omega)$*-Lebesgue points of* f.

The a.e. convergence for $L_1 \log L(\mathbb{T})$ and for Fejér means was proved by Marcinkiewicz (1939) and for $L_1(\mathbb{T})$ by Zhizhiashvili (1996).

Triangular summability

The triangular θ-means are defined by

$$\sigma_T^{1,\theta} f(x,y) := \frac{1}{2\pi} \int_{\mathbb{R}^2} \theta\left(\frac{|s|+|t|}{T}\right) \widehat{f}(s,t) e^{\imath(xs+yt)}\, ds\, dt.$$

Theorem (Weisz, 2018)

If $f \in W(L_1,\ell_\infty)(\mathbb{R}^2)$ and $0 < \omega < \min(\alpha,1)/2$, then

$$\lim_{T\to\infty} \sigma_T^{1,\theta} f(x,y) = f(x,y)$$

for all strong $(1,\omega)$-Lebesgue points of f.

The a.e. convergence for $L_1(\mathbb{R})$ and for Riesz means was proved by Berens, Li and Xu (2001).

Strong summability

Recall that

$$\sigma_T^\theta f(x) = \frac{-1}{T} \int_0^\infty \theta'\left(\frac{t}{T}\right) s_t f(x)\, dt.$$

Hence

$$\sigma_T^\theta f(x) - f(x) = \frac{-1}{T} \int_0^\infty \theta'\left(\frac{t}{T}\right) \Big(s_t f(x) - f(x)\Big)\, dt \to 0$$

as $T \to \infty$ at all 1-Lebesgue points if $f \in W(L_1, \ell_q)(\mathbb{R})$ for some $1 \leq q < \infty$. Is it true that

$$\frac{1}{T} \int_0^\infty \left|\theta'\left(\frac{t}{T}\right) \Big(s_t f(x) - f(x)\Big)\right|\, dt \to 0$$

a.e. or at Lebesgue points?

Theorem (Weisz, 2015)

Suppose that $f \in W(L_1, \ell_q)(\mathbb{R})$ *for some* $1 \leq q < \infty$ *and* $0 < r < \infty$. *If* x *is a 1-Lebesgue point of* f *and* f *is locally bounded at* x, *then*

$$\lim_{T \to \infty} \frac{-1}{T} \int_0^\infty \theta' \left(\frac{t}{T} \right) |s_t f(x) - f(x)|^r \, dt = 0.$$

Theorem (Hardy, Littlewood, 1935, Weisz, 2016)

Suppose that $f \in W(L_p, \ell_q)(\mathbb{R})$ *for some* $1 < p < \infty$, $1 \leq q < \infty$ *and* $0 < r < \infty$. *If* x *is a* p*-Lebesgue point of* f, *then*

$$\lim_{T \to \infty} \frac{-1}{T} \int_0^\infty \theta' \left(\frac{t}{T} \right) |s_t f(x) - f(x)|^r \, dt = 0.$$

If x is a 1-Lebesgue point of $f \in W(L_1, \ell_\infty)(\mathbb{R})$, then

$$\lim_{h \to 0} \frac{1}{h} \int_0^h |f(x-s) - f(x)| \, ds = 0$$

and so, for every fixed integer i,

$$\lim_{T \to \infty} \frac{T}{i} \int_{(i-1)/T}^{i/T} |f(x-s) - f(x)| \, ds = 0.$$

A point x is called a *Gabisoniya point* of f if for all $1 < \gamma < \infty$,

$$\lim_{T \to \infty} \sum_{i=1}^{\infty} \left(\frac{T}{i} \int_{(i-1)/T}^{i/T} |f(x-s) - f(x)| \, ds \right)^{\gamma} = 0.$$

Theorem (Gabisonya, 1973, Weisz, 2015)

Every Gabisoniya point is a 1-Lebesgue point and almost every point $x \in \mathbb{R}$ is a Gabisoniya point of $f \in W(L_1, \ell_\infty)(\mathbb{R})$.

Theorem (Gabisonya, 1973, Weisz, 2015)

Suppose that $f \in W(L_1, \ell_q)(\mathbb{R})$ for some $1 \leq q < \infty$ and $0 < r < \infty$. If x is a Gabisoniya point of f, then

$$\lim_{T\to\infty} \frac{-1}{T} \int_0^\infty \theta'\left(\frac{t}{T}\right) |s_t f(x) - f(x)|^r \, dt = 0.$$

Theorem (Gabisonya, 1973, Weisz, 2015)

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Remark that $W(L_1, \ell_q)(\mathbb{R}) \supset L_q(\mathbb{R})$ for all $1 \leq q < \infty$.

Example (Fejér summation)

Let

$$\theta(t) = \begin{cases} 1-|t|, & \text{if } |t| \le 1; \\ 0, & \text{if } |t| > 1. \end{cases}$$

Example (de La Vallée-Poussin summation)

Let

$$\theta(t) = \begin{cases} 1, & \text{if } |t| \le 1/2; \\ -2|t|+2, & \text{if } 1/2 < |t| \le 1; \\ 0, & \text{if } |t| > 1. \end{cases}$$

Example (Jackson-de La Vallée-Poussin summation)

Let

$$\theta(t) = \begin{cases} 1-3t^2/2+3|t|^3/4, & \text{if } |t| \le 1; \\ (2-|t|)^3/4, & \text{if } 1 < |t| \le 2; \\ 0, & \text{if } |t| > 2. \end{cases}$$

Example (Rogosinski summation)

Let

$$\theta(t) = \begin{cases} \cos \pi t/2, & \text{if } |t| \le 1+2j; \\ 0, & \text{if } |t| > 1+2j; \end{cases} \qquad (j \in \mathbb{N}).$$

Example (Weierstrass summation)

Let $\theta(t) = e^{-|t|^\gamma}$ for some $1 \le \gamma < \infty$. Note that if $\gamma = 1$, then we obtain the **Abel summation**.

Example (Picard and Bessel summations)

$\theta(t) = (1+|t|^\gamma)^{-\delta}$ $(0 < \delta < \infty, 1 \le \gamma < \infty, \gamma\delta > 1)$.

Example (Riesz summation)

Let

$$\theta(t) = \begin{cases} (1-|t|^\gamma)^\delta, & \text{if } |t| \le 1; \\ 0, & \text{if } |t| > 1. \end{cases}$$