RESULTS ON THE INVERSE SPECTRAL THEORY FOR THE S-L OPERATOR

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A motivation

Starting from the 60's, Gel'fand-Levitan-Marchenko et al. developed the scattering theory for the Schrödinger operator

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with potential $q(x)$.
Later, Dubrovin, Matveev, Novikov studied the inverse problem for the so-called algebro geometric potentials.
Both gave rise to a large amount of results concerning both the spectral theory of the Schrödinger operator and the connection between these potentials and the solution of the K-dV equation

$$u_t = 3uu_x - \frac{1}{2}u_{xxx}.$$

Connection with nonlinear evolution equations

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$$\begin{cases} -\varphi'' + q\varphi = \lambda\varphi & \text{Gel'fand, Levitan, Marchenko} \\ \\ u_t = 3uu_x - \frac{1}{2}u_{xxx} + cu_x, & \text{Gardner, Green, Kruskal, Miura} \end{cases}$$

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$$\begin{cases} -\varphi'' + \varphi = \lambda y\varphi, & \text{Beals, Sattinger, Constantin} \\ \\ u_t = y_x u + 2yu_x + 2cu_x, & \text{Constantin (Liouville transform)} \\ y = u - \frac{1}{2}u_{xx} \end{cases}$$

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It is worth developing a scattering theory for the operator defined by the eigenvalue equation $-\varphi'' + \varphi = \lambda y(x)\varphi$.

The scattering for the Schrödinger operator

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When $\mathcal{S}$ is defined on the half-lines $\mathbb{R}^{\pm}$, one defines transformation operators $K_{\pm}(x,t)$ in such a way that a solution $f_{\pm}(x,k)$ of the eigenvalue equation

$$-\varphi'' + q\varphi = k^2\varphi$$

with condition

$$f_{\pm}(x,k) \sim e^{\pm ikx}, \text{ as } x \to \pm\infty,$$

can be expressed as

$$f_{\pm}(x,k) = e^{\pm ikx} \pm \int_x^{\pm\infty} K_{\pm}(x,t)e^{\pm ikt}dt.$$

Note that $K_{\pm}(x,t)$ does not depend on k.

This problem is in turn equivalent to that of solving the PDEs

$$\begin{cases} K_{\pm,xx} - K_{\pm,tt} = q(x)K_{\pm} \\ \mp 2\dfrac{d}{dx}K_{\pm}(x,x) = q(x). \end{cases}$$

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The kernels $K_\pm(x,t)$ can be found under the basic Scattering assumptions

$$\int_x^\infty |x| \cdot |q(x)| dx < \infty, \text{ for } K_+$$

$$\int_{-\infty}^x (1+|x|)|q(x) dx < \infty, \text{ for } K_-.$$

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On the whole line, we look for solutions

$$\varphi_+(x, k) \sim \begin{cases} e^{ikx} + a_{12}(k)e^{-ikx}, & x \to -\infty \\ a_{11}(k)e^{ikx}, & x \to +\infty, \end{cases}$$

and

$$\varphi_-(x, k) =\sim \begin{cases} a_{22}(k)e^{-ikx}, & x \to -\infty \\ e^{-ikx} + a_{21}(k)e^{ikx}, & x \to +\infty. \end{cases}$$

The matrix $S(k) = (a_{ij}(k))$ is the Scattering matrix

The operator $\mathcal{S}$ can have at most a finite number of negative (simple)eigenvalues, namely $k_j^2 = -\chi_j^2$, hence $k_j = i\chi_j$ $(j = 1, \ldots, n)$.

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They coincide with the zeros of the function

$$b_{12} = \frac{1}{a_{22}}$$

The values

$$-i\dot{b}_{12} = M_j^+$$

are the norming constants

$$\int_{\mathbb{R}} f_+^2(i\chi_j, x)dx$$

Define

$$\Omega_+(t) = C_+(t) + \sum_{j=1}^{n} \frac{e^{-2\chi_j t}}{M_j^+}$$

$$A_+(x,t) + \Omega_+(x+t) + \int_0^\infty A_+(x,s)\Omega_+(x+t+s)ds = 0,$$

where

$$K_+(x,t) = \frac{1}{2} A_+\left(x, \frac{t-x}{2}\right)$$

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The algebro-geometric theory arises in the case of the so-called finite gap potentials, which are such that the spectrum of the full-line operators $\mathcal{S}$ is given by

$$\Sigma = [\lambda_0, \lambda_1] \cup [\lambda_2, \lambda_3] \cup \cdots \cup [\lambda_{2g}, \infty).$$

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In some cases these potentials define a meromorphic function $M(x, \lambda)$ ($k^2 = \lambda$) on an algebraic Riemann surface $\mathcal{R}$ of degree g, where one can study the properties of the potential itself

It turns out that this function $M(x, \lambda)$ has g simple poles $P_1(x), \ldots, P_g(x)$ which lie in the "holes" of the surface $\mathcal{R}$, and which determine the potential $q(x)$ via the trace formula

$$q(x) = \sum_{j=1}^{2g} \lambda_j - 2 \sum_{j=1}^{g} P_j(x).$$

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In this case, it turns out that $P_j(x)$ satisfy a system of g ODEs with initial condition $P_1(0), \ldots, P_g(0)$, the so-called pole motion. This defines $P_j(x)$ and hence $q(x)$.

$$P_j'(x) = \frac{2(-1)^g k(P_j(x))}{\prod(P_j(x) - P_i(x))},$$

$$k(\lambda) = \sqrt{-(\lambda - \lambda_0)(\lambda - \lambda_1) \ldots (\lambda - \lambda_{2g})}.$$

The pole motion linearizes on the Riemann surface $\mathcal{R}$

The scattering theory for the Sturm-Liouville operator

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$$\frac{1}{y(x)}\left(-D(p(x)D)+q(x)\right)$$

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1) Can we develop a scattering theory? If so, which are the changes that have to be made? What kind of potentials are of scattering type?

2) Can we define and study algebro-geometric potentials? Which trace formula can be derived? Which motion and Riemann surface $\mathcal{R}$ has to be considered?

For simplicity, we will assume $p(x) \equiv 1$

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We study the full line operator

$$\mathcal{SL} := \frac{1}{y(x)} \left(-D^2 + q(x) \right),$$

with domain $L^2(\mathbb{R}, y(x)dx)$, where $q(x), y(x)$ are bounded uniformly continuous functions, and $y(x)$ is strictly positive ($y(x) > \delta > 0$) and of class $C^2(\mathbb{R})$

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On the set $\mathcal{A} = \{a := (q, y) : \mathbb{R} \to \mathbb{R}^2\}$ satisfying the above properties, we introduce the translation flow $\tau_t(a) := a(\cdot + t)$.

if $a_0 = (q_0, y_0)$ is fixed, the set $\mathcal{A}_0 = cls\ Hull(a_0) = cls\{\tau_t(a_0)\ |t \in \mathbb{R}\}$ is a compact translation-invariant set

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The spectral problem associated to $\mathcal{SL}$ is

$$\mathcal{E} := -\varphi'' + q(x)\varphi = \lambda y(x)\varphi,$$

or

$$\begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ q(x) - \lambda y(x) & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}$$

It makes sense to define, for $a \in \mathcal{A}$, $a(\lambda, x) = \begin{pmatrix} 0 & 1 \\ q(x) - \lambda y(x) & 0 \end{pmatrix}$, and study the family

$$\begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}' = \tau_x(a) \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}$$

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Exponential dichotomy

The family has an exponential dichotomy over $\mathcal{A}$ if if there are positive constants η, ρ, together with a continuous, projection valued function $P : \mathcal{A} \to \mathbb{M}_2(\mathbb{C})$ such that the following estimates holds:

(I) $|\Phi_a(x)P(a)\Phi_a(s)^{-1}| \leq \eta e^{-\rho(x-s)}, \qquad x \geq s,$

(II) $|\Phi_a(x)(I - P(a))\Phi_a(s)^{-1}| \leq \eta e^{\rho(x-s)}, \qquad x \leq s.$

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$KerP(a)$ and $ImP(a)$ are complex lines in $\mathbb{C}^2$

we parametrize $KerP(a)$ and $ImP(a)$ by

$$ImP(a) = Span \begin{pmatrix} 1 \\ m_+(a,\lambda) \end{pmatrix} \quad KerP(a) = Span \begin{pmatrix} 1 \\ m_-(a,\lambda) \end{pmatrix}$$

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Under the flow, they give rise to $m_\pm(x,\lambda) := m_\pm(\tau_x(a),\lambda)$

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Take a local parameter at $\lambda = \infty$, $z^2 = -\lambda$, then

$$m_+(z) = \sqrt{y}z - \frac{y'}{4y} + \sum_{n=1}^{\infty} \alpha_{-n} z^{-n}$$

$$m_-(z) = -\sqrt{y}z - \frac{y'}{4y} + \sum_{n=1}^{\infty} (-1)^n \alpha_{-n} z^{-n}.$$

$\alpha_0 = -\frac{y'}{4y}$,

$$\alpha_{-1} = \frac{1}{2}\left[\frac{q - \alpha_0' - \alpha_0^2}{\sqrt{y}}\right]$$

and so on

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We want to develop an inverse scattering theory for $\mathcal{E}$ is the sense that we want to understand which are the scattering data and the equations of the inverse scattering which permit one to reconstruct $q(x)$ (resp. $y(x)$) when $y(x)$ (resp. $q(x)$) is fixed.

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Suppose now that the full line spectrum of $\mathcal{S}L$ has as a.c. part the half-line $[\lambda_*, +\infty)$.

Then set

$$\tilde{q} = -\left(\frac{y'}{4y}\right)' + \left(\frac{y'}{4y}\right)^2.$$

Set

$$\mu = \lambda - \lambda_*$$

and rewrite $\mathcal{E}$ as

$$-\varphi'' + \tilde{q}(x)\varphi - \mu y(x)\varphi = [\tilde{q}(x) - q(x) + \lambda_* y(x)]\varphi$$

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Finally, set

$$\mathcal{I}(x) = \int_0^x \sqrt{y(s)}ds.$$

The general solution of the equation

$$-\varphi'' + \tilde{q}(x)\varphi = \mu y(x)\varphi$$

is, for $\mu \geq 0$,

$$f(x,\mu) = (y(x))^{-1/4}\left(A \sin\sqrt{\mu}\mathcal{I}(x) + B\cos\sqrt{\mu}\mathcal{I}(x)\right).$$

The corresponding operator $\tilde{\mathcal{S}L}$ has spectrum $\tilde{\Sigma} = [0,\infty)$ and no isolated eigenvalues (recall that $\mu = \lambda - \lambda_*$)

Consider the half-line restricted operators $\mathcal{SL}^{\pm}$, together with the condition

$$\varphi_{\pm}(x,\mu) \sim y(x)^{-1/4} e^{\pm i\sqrt{\mu}\mathcal{I}(x)}, \quad x \to \pm\infty.$$

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$$\varphi_{\pm}(x,\mu) \sim y(x)^{-1/4} e^{\pm i\sqrt{\mu}\mathcal{I}(x)}, \quad x \to \pm\infty.$$

We write

$$\varphi_{\pm}(x,\mu) = y(x)^{-1/4} e^{\pm i\sqrt{\mu}\mathcal{I}(x)} \pm \int_x^{\pm\infty} K_{\pm}(x,t) y(t) \left[y(t)^{-1/4} e^{\pm i\sqrt{\mu}\mathcal{I}(t)} \right] dt,$$

and look for the existence of $K_{\pm}(x,t)$.

$K_{\pm}(x,t)$ is defined in a domain $\mathcal{D}_{\pm} = \{(x,t) \mid t \geq (\leq) x\}$, satisfies some specific regularity conditions, and

$$\lim_{x+t\to\pm\infty} \max\{K_{\pm}, K_{\pm,x}, K_{\pm,t}\} = 0.$$

It turns out that $K_\pm$ satisfy

$$\begin{cases} \dfrac{1}{y(x)} K_{\pm,xx}(x,t) - \dfrac{1}{y(t)} K_{\pm,tt}(x,t) = \left[\dfrac{q(x)}{y(x)} - \dfrac{\tilde{q}(t)}{y(t)} - \lambda_* \right] K_\pm(x,t) \\ \\ \pm 2y(x) \dfrac{d}{dx} K_\pm(x,x) \pm K_\pm(x,x) \dfrac{d}{dx} y(x) = \tilde{q}(x) - q(x) + \lambda_* y(x). \end{cases}$$

By using the variation of constants formula, and writing $k^2 = \mu$ (for notational and computational convenience), we express solutions of $\mathcal{E}$ on the half-lines satisfying

$$\varphi_\pm(k,x) \sim y^{-1/4}(x) e^{\pm ik\mathcal{I}(x)}, \quad x \to \pm\infty,$$

as

$$\varphi_\pm(k,x) = y^{-1/4}(x)\Big[e^{\pm ik\mathcal{I}(x)} \pm \int_0^{\pm\infty} \frac{\sin(\pm k(\mathcal{I}(x)-\mathcal{I}(t))}{k} \cdot$$

$$[\tilde{q}(t) - q(t) + \lambda_* y(t)] y^{-1/4}(t) \varphi_\pm(k,t) dt\Big].$$

These equations can be solved by successive approximations,

BUT...

The Scattering Hypothesis on the line

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$$\int_{\mathbb{R}} (1 + |\mathcal{I}(x)|) \cdot |\tilde{q}(x) - q(x) + \lambda_* y(x)| dx < +\infty.$$

The solutions $\varphi_{\pm}(k, x)$ are analytic for $\Im k > 0$ and continuous and bounded for $\Im k = 0$ for all $x \in \mathbb{R}$

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If α is a real number,

$$\int_{\mathbb{R}} |y^{-1/4}(x)\varphi_{\pm}(\alpha + i\varepsilon, x) - e^{\pm i(\alpha+i\varepsilon)\mathcal{I}(x)}|^2 d\alpha = \mathcal{O}\left(e^{-2|\varepsilon|\mathcal{I}(x)}\right)$$

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We can infer that there exist $B_{\pm}(x,t)$ such that

$$\varphi_{\pm}(k,x) = y^{-1/4}(x)\left[e^{\pm ik\mathcal{I}(x)} \pm \int_{\mathcal{I}(x)}^{\pm\infty} B_{\pm}(x,t)e^{\pm ikt}dt\right],$$

$$K_{\pm}(x,t) = y^{-1/4}(x)B_{\pm}(x,\mathcal{I}(t))y^{-1/4}(t).$$

The scattering problem on the whole line

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Let the Scattering Hypothesis be valid

$$\int_{\mathbb{R}} (1 + |\mathcal{I}(x))|\tilde{q}(x) - q(x) + \lambda_* y(x)|dx < \infty.$$

On the whole line, we look for solutions (k real)

$$\varphi_+(x,k) \sim \begin{cases} y^{-1/4}(x)[e^{ik\mathcal{I}(x)} + a_{12}(k)e^{-ik\mathcal{I}(x)}], & x \to -\infty \\ a_{11}(k)y^{-1/4}(x)e^{ik\mathcal{I}(x)}, & x \to +\infty, \end{cases}$$

and

$$\varphi_-(x,k) =\sim \begin{cases} a_{22}(k)y^{-1/4}(x)e^{-ik\mathcal{I}(x)}, & x \to -\infty \\ y^{-1/4}(x)[e^{-ik\mathcal{I}(x)} + a_{21}(k)e^{ik\mathcal{I}(x)}], & x \to +\infty. \end{cases}$$

a_{12}, a_{21} are reflection and a_{11}, a_{22} are transmission coefficients

The coefficients a_{11} and a_{21} determine uniquely the other coefficients as well

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The function $b_{12}(k) = \dfrac{1}{a_{11}(k)}$ admits an analytic extension to the upper k-plane. The zeros of $b_{12}(k)$ must be of the form $k_j = i\chi_j$ and are simple and finite. The quantities $k_j^2 = -\chi_j^2$ are the isolated negative eigenvalues of $\mathcal{SL}$ on the whole line.

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For each eigenvalue $-\chi_j^2$, the corresponding solutions $\varphi_\pm(x, i\chi_j)$ are such that $\varphi_+(x, i\chi_j) = c_j\varphi_-(x, i\chi_j)$, hence the norming constants M_j^+ and M_j^- are given.

We have all the information needed to solve the inverse problem

The scattering data is the set $\{a_{12}(k), \chi_1, \ldots, \chi_n, M_1^+, \ldots, M_n^+\}$ where $k \in \mathbb{R}$.

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Set

$$a_{11}(k) = \exp\left[\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\ln(1-|a_{12}(\rho)|^2)^{1/2}}{\rho - k} d\rho\right] \prod_{i=1}^{n} \frac{k + i\chi_i}{k - i\chi_i}$$

$b_{22}(k) = a_{12}(k)a_{11}(k)$ and $a_{21}(k) = -b_{22}(-k)a_{11}(k)$

$$a_{12}(k) = \int_{\mathbb{R}} C_-(t)e^{2ikt}dt, \qquad a_{21}(k) = \int_{\mathbb{R}} C_+(t)e^{-2ikt}dt.$$

$$M_j^- M_j^+ = \frac{-1}{[Res\ a_{11}(k)_{k=i\chi_j}]^2}$$

$$\Omega_\pm(t) = C_\pm(t) + \sum_{j=1}^{n} \frac{e^{\mp 2\chi_j t}}{M_j^\pm}$$

Let $A_{\pm}(x,t)$ be the solution of the equations

$$A_{\pm}(x,t)+\Omega_{\pm}(\mathcal{I}(x)+t)\pm y^{-1/4}(x)\int_0^{\pm\infty}A_{\pm}(x,s)\Omega_{\pm}(\mathcal{I}(x)+t+s)ds=0.$$

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$A_+(x,t)$ is defined for $t>0$ and $A_-(x,t)$ for $t<0$

$$K_{\pm}(x,t)=\frac{1}{2}y^{-1/4}(x)A_{\pm}\left(x,\frac{\mathcal{I}(t)-\mathcal{I}(x)}{2}\right)y^{-1/4}(t)$$

$K_+(x,t)$ defined for $t>x$ and $K_-(x,t)$ for $t<x$

For fixed $y(x)$ (resp. $q(x)$) retrieve two functions $q_\pm(x)$ (resp. $y_\pm(x)$) via the relation

$$\pm 2y(x)\frac{d}{dx}K_\pm(x,x) \pm K_\pm(x,x)\frac{d}{dx}y(x) = \tilde{q}(x) - q(x) + \lambda_* y(x)$$

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Then $q_+(x) = q_-(x)$ (resp. $y_+(x) = y_-(x)$) and $K_\pm(x,t)$ satisfy the equations

$$\frac{1}{y(x)}K_{\pm,xx}(x,t) - \frac{1}{y(t)}K_{\pm,tt}(x,t) = \left[\frac{q(x)}{y(x)} - \frac{\tilde{q}(t)}{y(t)} - \lambda_* y(x)\right] K_\pm(x,t).$$

The Algebro-geometric inverse problem

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Assume, for simplicity, that $\lambda = 0$ is not an eigenvalue

Recall the Weyl m-functions $m_{\pm}(x, \lambda)$ defined via the exponential dichotomy.

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Recall the Weyl m-functions $m_\pm(x, \lambda)$ defined via the exponential dichotomy.

Since $\lambda = 0$ is not an eigenvalue, the functions $m_\pm(x, 0)$ are well-defined. Set

$$\mathcal{M}(x) = m_-(x, 0) - m_+(x, 0)$$

We assume that:

1) The spectrum Σ of $\mathcal{SL}$ is a finite union of intervals, plus a half-line:

$$\Sigma = [\lambda_0, \lambda_1] \cup [\lambda_2, \lambda_3] \cup \cdots \cup [\lambda_{2g}, \infty).$$

2) The Lyapunov exponent $\beta(\lambda)$ vanishes a.e. on Σ

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1) The spectrum Σ of $\mathcal{SL}$ is a finite union of intervals, plus a half-line:

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2) The Lyapunov exponent $\beta(\lambda)$ vanishes a.e. on Σ

The functions $m_\pm(x, \lambda)$ extend holomorphically through every open interval I contained in Σ. Both are meromorphic functions having poles $P_1(x), \ldots, P_g(x)$ in the spectral gaps $[\lambda_1, \lambda_2], \ldots, [\lambda_{2g-1}, \lambda_{2g}]$, with the convention that each $P_j(x)$ is a pole of either m_- or m_+

The values $P_j(x)$ are the isolated eigenvalues of the half-line restricted operators (when they do not coincide with the endpoints of the spectral gaps)

The inverse spectral theory in this case is to determine which spectral parameters are necessary and sufficient to determine $q(x)$ and $y(x)$. In this case we choose as scattering set the quantities

$$\{\mathcal{M}(x), P_1(0), \dots, P_g(0)\}$$

Let $P_j(x)$ satisfy the system of ODEs

$$P_j'(x) = \frac{(-1)^g k(P_r(x))\mathcal{M}(x) \prod P_i(x)}{k(0) \prod (P_j(x) - P_i(x))},$$

where $k(\lambda) = \sqrt{-(\lambda - \lambda_0) \dots (\lambda - \lambda_{2g})}$

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$$2\sqrt{y(x)} = \frac{(-1)^{g+1}\mathcal{M}(x)\prod P_i(x)}{k(0)}$$

$$q(x) = y(x)\left(\sum_{i=0}^{2g}\lambda_i - 2\sum_{i=1}^{g}P_i(x)\right) + \tilde{q}(x).$$

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The operator $\mathcal{SL}$ with potentials $q(x)$ and $y(x)$ defined as above has spectrum $\Sigma = [\lambda_0, \lambda_1] \cup \dots [\lambda_{2g}, \infty)$ and vanishing Lyapunov exponent $\beta(\lambda)$ for a.a. $\lambda \in \Sigma$

Evolution equations of shallow water waves

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If $u_0(x) := u(x,0) = q(x)$, then there is an associated Schrödinger equation

$$-\varphi'' + q(x)\varphi = k^2\varphi.$$

Assume that $q(x)$ can be recovered via the scattering data , i.e., via the coefficient $b(k) := a_{12}(k)$, the discrete spectrum $-\chi_j^2$ $(j = 1, \ldots, n)$ and the norming constant which we assume to be all equal to 1. Take now the coefficient $a_{11}(k) =: a(k)$

It turns out that under the K-dV flow, the solution $u(x,t)$ starting from $u_0(x)$ defines potentials $q(x,t)$ and corresponding eigenvalue equations

$$-\varphi'' + q(x,t)\varphi = k^2\varphi,$$

such that the isolated eigenvalues $-\chi_j^2(t)$ do not vary w.r.t. $t > 0$: $-\chi_j^2(t) = \chi_j^2(0) =: E_j$ for all $j = 1, \dots, n$.

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χ_j are determined from $u_0(x)$

The other quantities $a(k, t)$ and $b(k, t)$ evolve according to

$$a(k,t) = a(k,0)$$

$$b(k,t) = b(k,0)e^{8ik^3t}$$

$$\begin{cases} y = 2u - \dfrac{1}{2}u_{xx} \\ \\ y_t = y_x u + 2yu_x \end{cases} \qquad (Camassa - Holm)$$

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We start from an initial condition $u_0(x) := u(x,0)$ which in turn defines $y(x) = 2u_0(x) - \frac{1}{2}u_{0,xx}(x)$.

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The associated eigenvalue equation is in this case

$$-\varphi'' + \varphi = k^2 y(x)\varphi.$$

Under the scattering hypothesis, we assume that $y(x)$ satisfies

$$\int_{\mathbb{R}} (1 + |\mathcal{I}(x)|) \cdot |\tilde{q}(x) - 1 + \lambda_* y(x)| dx < \infty,$$

which means that, if $y(x)$ decays rapidly to some constant, it decays to $\dfrac{1}{\lambda_*}$. λ_* is also the left endpoint of the a.c. spectrum of the operator.

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With $\mu = \lambda - \lambda_*$ and $k^2 = \mu$, we reconstruct $y(x)$ via the inverse scattering theory, defining the coefficients $a(k), b(k)$ and the isolated eigenvalues $-\chi_j^2 = \mu_j = \lambda_j - \lambda_*$

If $u(x,t)$ evolves according the Camassa-Holm equation, then there is a family of potentials $y(x,t) = 2u(x,t) - \frac{1}{2}u_{xx}(x,t)$ which defines a family of equations

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Again, it turns out that the discrete eigenvalues $-\chi_j^2$ do not depend on $t > 0$.

The remaining coefficients evolve according to

$$a(k,t) = a(k,0)$$

$$b(k,t) = b(k,0)\exp\left(\frac{ik}{k^2+\lambda_*}\right).$$

As another example, the system of equations (α is a function which satisfies some prescribed properties)

$$\begin{cases} q_t = \alpha q_x + 2\alpha_x q - \dfrac{\alpha_{xxx}}{2} \\ \\ u_t = \alpha u_x - 2\alpha_x u - \dfrac{q_x}{\sqrt{y}} + 2qu^3 u_x + u^3 u_{xxx} \\ \\ \sqrt{y} = 1/u \end{cases}$$

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is linked to a general equation $-\varphi'' + q(x)\varphi = \lambda y(x)\varphi$.

Again, the flow induced by the solution $y(t,x)$ and the evolution of $q(x,t)$ defines an isospectral flow, i.e, the spectrum of the equation $-\varphi'' + q(x,t)\varphi = \lambda y(x,t)\varphi$ does not depend on $t > 0$

The determination of the solution $y(x,t)$ can be then carried out by studying the evolution of the coefficients $a(k,t)$ and $b(k,t)$ as before.

The above example is a case of a much more general situation in which there is a hierarchy of nonlinear evolution equations which includes the K-dV equation and the Camassa-Holm equations as special cases, and the spectrum of a (general) Sturm-Liouville operator associated to the solutions of these hierarchies, is invariant when the solution starts from some families of initial conditions. We called it the **Sturm-Liouville hierarchy** of evolution equations. It has applications in the dynamics of magnetic fields and light scattering...

... This is a work in progress...

Spotted stains in rainbows

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It happens rarely, but sometimes spotted stains of different colors appear in a certain color band in a rainbow. Practically, sometimes in the red strip of a rainbow one can see spots of purple color, which can be either distributed on a long line, or concentrated on small area

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A way to represent mathematically this phenomenon it that of perturbing the energy λ by a weight y.

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One looks at the (radial) equation

$$-\varphi'' + \frac{2mV_0}{\hbar}\varphi = k^2\varphi.$$

$$N = \left(1 + \frac{2mV_0}{\hbar k^2}\right).$$

If k is fixed, $N > 1$ corresponds to a potential well, while $N < 1$ corresponds to a barrier. N is the refraction index, and $-V_0$ is the square well depth.

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What happens if

$$N = N(r) = \left(y(r) + \frac{2mV(r)}{\hbar k^2}\right)?$$

The governing equation is now

$$-\varphi'' + \frac{2mV(r)}{\hbar}\varphi = k^2 y(r)\varphi.$$