Series representation of integral kernels of transmutation operators and applications to numerical solution of spectral problems

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In the talk a new representation for the solution of an equation

$$-y'' + \frac{\ell(\ell+1)}{x^2}y + q(x)y = \omega^2 y, \qquad x \in (0, b] \tag{1}$$

(known as the one-dimentional Schrödinger equation when $\ell = 0$ and the perturbed Bessel equation when $\ell \neq 0$, $\ell \geq -1/2$) in the form of Neumann series of Bessel functions

$$y_\ell(\omega, x) = b_\ell(\omega x) + \sum_{n=0}^{\infty} \beta_n(x) j_{2n+\ell+\delta}(\omega x)$$

is constructed. Here b_ℓ is a solution of the unperturbed equation (having $q \equiv 0$), $\delta = 1$ for $\ell \neq 0$ and can be either 0 or 1 for $\ell = 0$ and j_z are the spherical Bessel functions.

Some ideas behind such representation:

- existence of an integral representation for a solution $y_\ell(\omega, x)$ of the perturbed equation of the form

$$y_\ell(\omega, x) = b_\ell(\omega, x) + \int_{-x}^{x} K(x,t) d_\ell(\omega, t)\, dt,$$

  here b_ℓ is the solution of the unperturbed equation and d_ℓ is a known function.
- Transmutation operators or asymptotic estimates and Paley-Wiener theorem as sources of such integral representations.
- Expansion of the integral kernel K into Fourier-Legendre (or Fourier-Jacobi) series with respect to the variable t.
- Convenient formulas and decay rate estimates for the expansion coefficients.

Suppose that we approximated the integral kernel K by a function K_N,

$$||K(x,\cdot) - K_N(x,\cdot)||_{L_2(-x,x)} \le \varepsilon(x), \qquad 0 < x \le b,$$

and define as an approximate solution

$$y_N(\omega, x) = b_\ell(\omega, x) + \int_{-x}^{x} K_N(x,t) d_\ell(\omega, t)\, dt.$$

Then for all ω from some (complex) region G

$$\begin{aligned} |y_\ell(\omega,x) - y_N(\omega,x)|^2 &\le \left(\int_{-x}^{x} |K(x,t) - K_N(x,t)| \cdot |d_\ell(\omega,t)|\, dt\right)^2 \\ &\le \int_{-x}^{x} |K(x,t) - K_N(x,t)|^2\, dt \cdot \int_{-x}^{x} |d_\ell(\omega,t)|^2\, dt \\ &\le \varepsilon^2(x) \int_{-x}^{x} |d_\ell(\omega,t)|^2\, dt \le C_G \varepsilon^2, \end{aligned}$$

provided the integral of the squared function d_ℓ is bounded in the region G.

Motivating example 1[1]: a problem from [J. D. Pryce, 1993]

$$\begin{cases} -u'' + e^x u = \lambda u, \\ u(0) = 0, \qquad u(\pi) = 0. \end{cases}$$

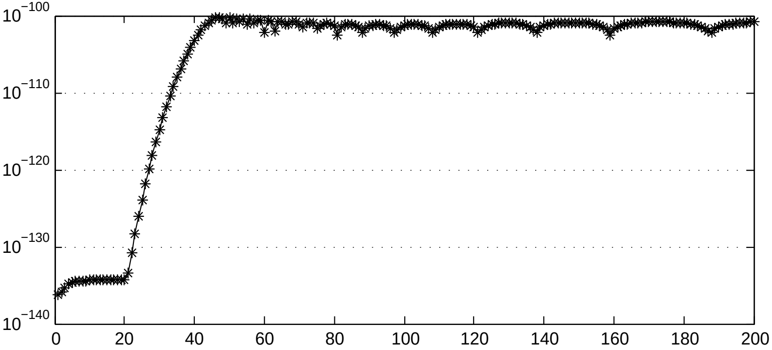

Figure: Absolute errors of the first 200 eigenvalues, high precision arithmetics and 73 terms of the Neumann series used. Computation time: 22s.

[1] Example from Kravchenko, Navarro, Torba, 2015, case $\ell = 0$

$$-u'' + \left(\frac{\ell(\ell+1)}{x^2} + x^2\right) u = \omega^2 u, \qquad u(\omega,\pi) = 0.$$

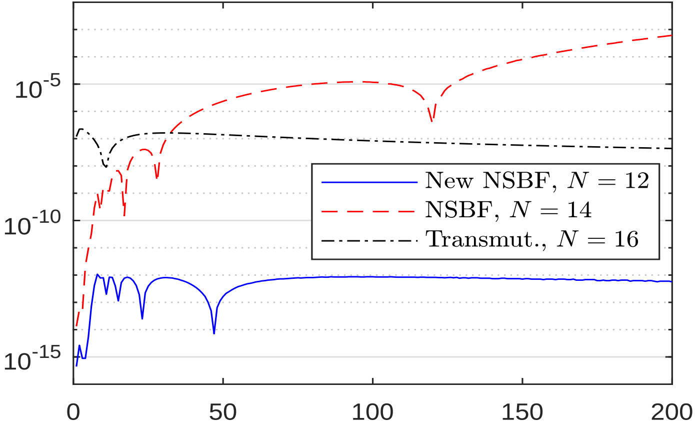

Figure: Case $\ell = 5$. Absolute errors of the first 200 eigenvalues. Computation time: 0.25s.

An operator T is called[2] a **transmutation (transformation) operator** for a pair of operators A and B if it is continuous and continuously invertible on a suitable topological space and satisfy the operator equality

$$AT = TB.$$

[2] B. M. Levitan, Inverse Sturm-Liouville problems, 1987

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Convenience of knowing a transmutation operator:

Let v be a solution of the equation $Bv = \omega^2 v$ and consider $u = Tv$. Then

$$Au = ATv = TBv = T[\omega^2 v] = \omega^2 u,$$

i.e., u is the solution of the more complicated equation $Au = \omega^2 u$.

[2] B. M. Levitan, Inverse Sturm-Liouville problems, 1987

The case $\ell = 0$ of one dimensional Schrödinger operators

Consider a pair of second order differential operators

$$A = -\frac{d^2}{dx^2} + q(x) \quad \text{and} \quad B = -\frac{d^2}{dx^2}.$$

It is sufficient to know the functions

$$T[\cos \omega x] \qquad \text{and} \qquad T[\sin \omega x].$$

[3] V. M. Marchenko, Some questions on the theory of one-dimensional linear second order differential operators, 1952 (in Russian)

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A family of transmutation operators (for a continuous q) on $C^2[-b, b]$ can be realized as Volterra integral operators[3]

$$T_h u(x) = u(x) + \int_{-x}^{x} K_h(x, t) u(t)\, dt,$$

where $h \in \mathbb{C}$,

[3] V. M. Marchenko, Some questions on the theory of one-dimensional linear second order differential operators, 1952 (in Russian)

The integral kernel of the transmutation operator

and the integral kernel $K_h(x,t)$ of T_h is the solution of the Goursat problem

$$\left(\frac{\partial^2}{\partial x^2} - q(x)\right) K_h(x,t) = \frac{\partial^2}{\partial t^2} K_h(x,t),$$

$$K_h(x,x) = \frac{h}{2} + \frac{1}{2}\int_0^x q(s)\,ds, \qquad K_h(x,-x) = \frac{h}{2}.$$

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Unfortunately, this Goursat problem can be exactly solved only in some particular cases. Even for the case $q = c^2$ the kernel of T_0 is

$$K_0(x,y) = -\frac{1}{2}\frac{\sqrt{c(x^2-y^2)}J_1\left(\sqrt{c(x^2-y^2)}\right)}{x-y}.$$

The mapping property of transmutation operators

Let $q : [0, b] \to \mathbb{C}$ and f be a particular non-vanishing solution of

$$f'' - qf = 0$$

such that $f(0) = 1$, and let $h := f'(0) \in \mathbb{C}$. Define $\widetilde{X}^{(0)} \equiv X^{(0)}(x) \equiv 1$,

$$\widetilde{X}^{(n)} = n \int_0^x \widetilde{X}^{(n-1)} \cdot f^{2(-1)^{n-1}}\,\mathrm{d}s, \quad X^{(n)} = n \int_0^x X^{(n-1)} \cdot f^{2(-1)^{n}}\,\mathrm{d}s,$$

$$\varphi_{2k} = f \cdot \widetilde{X}^{(2k)} \quad \text{and} \quad \varphi_{2k+1} = f \cdot X^{(2k+1)}.$$

Theorem ([Campos, Kravchenko, Torba, 2012])

The transmutation operator T_h maps x^k into $\varphi_k(x)$, i.e.,

$$\varphi_k(x) = T_h[x^k] = x^k + \int_{-x}^{x} K_h(x,t)t^k\,dt.$$

In particular, $T_h[1] = f$.

Corollary: scalar products

For each fixed $0 < x \le b$ we know the scalar products

$$\left\langle K_h(x,t), t^k \right\rangle_{L_2(-x,x)} = \int_{-x}^{x} K_h(x,t) t^k\, dt$$
$$= \varphi_k(x) - x^k, \qquad k = 0, 1, \ldots$$

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Legendre polynomials

$$P_n\left(\frac{t}{x}\right), \qquad n = 0, 1, \ldots$$

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Hence, we are looking for Fourier-Legendre series representation for K_h,

$$K_h(x,t) = \sum_{n=0}^{\infty} \alpha_n(x) P_n\left(\frac{t}{x}\right).$$

Theorem ([Kravchenko, Navarro, Torba, 2015])

Let $q \in C[0,b]$. Then the transmutation kernel K_h has the form

$$K_h(x,t) = \sum_{n=0}^{\infty} \frac{\beta_n(x)}{x} P_n\left(\frac{t}{x}\right)$$

where for every $x \in (0,b]$ the series converges uniformly wrt. $t \in [-x,x]$;

$$\beta_n(x) = \frac{2n+1}{2} \int_{-x}^{x} K(x,t) P_n\left(\frac{t}{x}\right)\, dt = \frac{2n+1}{2}\left(\sum_{k=0}^{n} \frac{l_{k,n}\varphi_k(x)}{x^k} - 1\right),$$

with $l_{k,n}$ being the coefficient at x^k of the Legendre polynomial P_n.

Representation of the solutions

Two linearly independent solutions of the equation

$$-u'' + q(x)u = \omega^2 u, \qquad x \in [0, b].$$

can be represented as follows

$$c(x,\omega) := T_h[\cos\omega x] = \cos\omega x + \int_{-x}^{x} K_h(x,t)\cos\omega t\,dt,$$

$$s(x,\omega) := T_h[\sin\omega x] = \sin\omega x + \int_{-x}^{x} K_h(x,t)\sin\omega t\,dt.$$

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Substituting the integral kernel representation we obtain that

$$c(x,\omega) = \cos\omega x + \sum_{j=0}^{\infty} \beta_j(x) \int_{-1}^{1} P_j(y)\cos(\omega x y)\ dy,$$

$$s(x,\omega) = \sin\omega x + \sum_{j=0}^{\infty} \beta_j(x) \int_{-1}^{1} P_j(y)\sin(\omega x y)\ dy.$$

Neumann series representations

The following integrals are known

$$\int_0^a \left\{ \begin{array}{l} P_{2n+1}\left(\frac{y}{a}\right)\cdot \sin by \\ P_{2n}\left(\frac{y}{a}\right)\cdot \cos by \end{array} \right\} dy = (-1)^n\, a\cdot j_{2n+\delta}(ab), \quad \delta = \left\{ \begin{array}{c} 1 \\ 0 \end{array} \right\}.$$

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Theorem ([Kravchenko, Navarro, Torba, 2015])

The solutions $c(\omega, x)$ and $s(\omega, x)$ admit the following representations

$$c(x,\omega) = \cos\omega x + 2\sum_{n=0}^{\infty}(-1)^n \beta_{2n}(x) j_{2n}(\omega x),$$

$$s(x,\omega) = \sin\omega x + 2\sum_{n=0}^{\infty}(-1)^n \beta_{2n+1}(x) j_{2n+1}(\omega x)$$

where j_k stands for the spherical Bessel function of order k, the series converge uniformly wrt. x on $[0, b]$ and wrt. ω on any compact subset of the complex plane.

Convergence rate of the partial sums

Consider the truncated series

$$c_N(x,\omega) = \cos\omega x + 2\sum_{n=0}^{N}(-1)^n\beta_{2n}(x)j_{2n}(\omega x),$$

$$s_N(x,\omega) = \sin\omega x + 2\sum_{n=0}^{N}(-1)^n\beta_{2n+1}(x)j_{2n+1}(\omega x).$$

Theorem

Let $q \in W_2^p[0,b]$. Then there exists a constant $c_p > 0$ such that

$$|c(x,\omega) - c_N(x,\omega)| \le \frac{c_p x^{p+3/2}}{N^{p+1}},$$

$$|s(x,\omega) - s_N(x,\omega)| \le \frac{c_p x^{p+3/2}}{N^{p+1}}.$$

for all $\omega \in \mathbb{R}$.

Perturbed Bessel equation ($\ell \neq 0$)

Consider the perturbed Bessel equation

$$-y'' + \frac{\ell(\ell+1)}{x^2}y + q(x)y = \omega^2 y, \qquad x \in (0, b], \quad \ell \geq -1/2.$$

- Transmutation operator for the perturbed Bessel equation is known[4] and acts as

$$y(\omega, x) = \mathcal{T}\left[b_\ell(\omega x)\right] := b_\ell(\omega x) + \int_0^x K(x,t) b_\ell(\omega t)\, dt,$$

where

$$b_\ell(\omega x) := \sqrt{\omega x} J_{\ell+\frac{1}{2}}(\omega x).$$

[4] V. Volk, On inversion formulas for a differential equation with a singularity at $x = 0$, Uspehi Matem. Nauk (N.S.), 1953

Perturbed Bessel equation

- The mapping property of the transmutation operator: let u_0 be a non-vanishing on $(0, b]$ solution of the equation

$$-u_0'' + \left(\frac{\ell(\ell+1)}{x^2} + q(x)\right) u_0 = 0$$

satisfying asymptotic $u_0(x) \sim x^{\ell+1}$, $x \to 0$. Define $\widetilde{X}^{(0)} \equiv 1$,

$$\widetilde{X}^{(n)}(x) = \int_0^x u_0^{2(-1)^{n-1}}(t)\widetilde{X}^{(n-1)}(t)\,dt, \qquad n \geq 1,$$

and finally $\varphi_{2n} := u_0 \cdot (2n)!\widetilde{X}^{(2n)}$. Then [5]

$$\mathcal{T}[x^{2k+\ell+1}] = \frac{(-1)^k 2^{2k} k!}{(2k)!}\left(\ell + \frac{3}{2}\right)_k \varphi_{2k}(x), \qquad k \in \mathbb{N}_0.$$

[5] R. Castillo-Perez, V. V. Kravchenko, S. M. Torba, Spectral parameter power series for perturbed Bessel equations, 2013

The previous scheme can be used for the case $\ell \neq 0$.

[6] V. V. Kravchenko, S. M. Torba and J. Yu. Santana-Bejarano, "Generalized wave polynomials and transmutations related to perturbed Bessel equations", 2019.

[7] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, "Integrals and series. Vol. 2. Special functions", 1986, 750 pp.

The previous scheme can be used for the case $\ell \neq 0$.

Some difficulties appear.

- Not much is known about the integral kernel K. E.g., one can not find any smoothness results required to establish decay estimates for the Fourier-Legendre series.
- The integrals

$$\int_0^x \sqrt{t} J_{\ell+1/2}(\omega t) P_n\left(\frac{t}{x}\right)\, dt$$

appearing are known only in the terms of generalized hypergeometric function ${}_2F_3$, less convenient for applications.[6] [7]

[6] V. V. Kravchenko, S. M. Torba and J. Yu. Santana-Bejarano, "Generalized wave polynomials and transmutations related to perturbed Bessel equations", 2019.

[7] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, "Integrals and series. Vol. 2. Special functions", 1986, 750 pp.

Another integral representation for the solution holds:

$$y_\ell(\omega,x) = d_\ell(\omega) b_\ell(\omega x) + \int_0^x R(x,t)\cos\omega t\,dt, \tag{2}$$

where

$$b_\ell(\omega x) := \sqrt{\omega x} J_{\ell+\frac12}(\omega x) \qquad \text{and} \qquad d_\ell(\omega) := \frac{2^{\ell+\frac12}\Gamma\left(\ell+\frac32\right)}{\omega^{\ell+1}}.$$

Follows from the asymptotic estimate[8]

$$|y_\ell(\omega,x) - d_\ell(\omega) b_\ell(\omega x)| \le C\left(\frac{x}{b+|\omega|x}\right)^{\ell+1} e^{|\operatorname{Im}\omega|x}\int_0^x \frac{t\tilde q(t)}{b+|\omega|t}\,dt$$

and the Paley-Wiener theorem.

[8] A. Kostenko, A. Sakhnovich, G. Teschl, Inverse eigenvalue problems for perturbed spherical Schrödinger operators, 2010

Neumann series representation of the regular solution

Expending the kernel R into Fourier-Legendre series with respect to t allowed us to obtain the following result.

Theorem (CKT, 2018)

The regular solution $y_\ell(\omega, x)$ has the form

$$y_\ell(\omega, x) = d_\ell(\omega) b_\ell(\omega x) + \sum_{n=0}^{\infty} (-1)^n \beta_n(x) j_{2n}(\omega x),$$

the series converges uniformly wrt. x on $[0, b]$ and wrt. ω on any compact subset of the complex plane.

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the series converges uniformly wrt. x on $[0, b]$ and wrt. ω on any compact subset of the complex plane. For the approximate solution

$$y_{\ell;N}(\omega, x) = d_\ell(\omega) b_\ell(\omega x) + \sum_{n=0}^{N} (-1)^n \beta_n(x) j_{2n}(\omega x)$$

Theorem

the following estimate holds

$$|y_\ell(\omega, x) - y_{\ell;N}(\omega, x)| \le x\varepsilon_N(x), \qquad \omega \in \mathbb{R}.$$

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Moreover, if $\ell \in \mathbb{N}$ *and* $q \in C^{2p-1}[0, b]$ *for some* $p \in \mathbb{N}$ *then*

$$\varepsilon_N(x) \le \frac{c_1 x^{\ell+p+3/2}}{N^{\ell+p+3/2}}, \qquad 2N \ge [\ell + p + 5/2],$$

and if $\ell \notin \mathbb{N}$ *and* $q \in C^{2p-1}[0, b]$ *for some* $p \in \mathbb{N}$ *then*

$$\varepsilon_N(x) \le \frac{c_2 x^{\ell+3/2}}{N^r}, \qquad N \ge [\ell + p + 3],$$

where $r = \min\{\ell + p + 1, 2\ell + 3\}$.

1. Uniform error bound for a decaying solution.
   Recall the first term: $d_\ell(\omega)b_\ell(\omega x)$. The function

$$b_\ell(\omega x) = \sqrt{\omega x} J_{\ell+\frac{1}{2}}(\omega x)$$

is uniformly bounded for $\omega \in \mathbb{R}$, while

$$d_\ell(\omega) := \frac{2^{\ell+\frac{1}{2}}\Gamma\left(\ell+\frac{3}{2}\right)}{\omega^{\ell+1}}$$

decays like $\omega^{-\ell-1}$ as $\omega \to \infty$.

Some disadvantages

1. Uniform error bound for a decaying solution.
   Recall the first term: $d_\ell(\omega)b_\ell(\omega x)$. The function

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   decays like $\omega^{-\ell-1}$ as $\omega \to \infty$.
2. Saturation phenomenon.
   For $\ell \notin \mathbb{N}$, independently on the smoothness of q we have

$$|\varepsilon_N(x)| \geq \frac{c_2 x^{\ell+2}}{N^{2\ell+3}}.$$

The integral kernels R and K are related by the following relations [9]

$$R(x,s) = \frac{2\Gamma\left(\ell+\frac{3}{2}\right)}{\sqrt{\pi}\Gamma(\ell+1)} \int_s^x K(x,t) t^{-\ell}(t^2-s^2)^\ell\, dt,$$

and

$$K(x,t) = \frac{2\sqrt{\pi}}{\Gamma\left(\ell+\frac{3}{2}\right)} \frac{t^{\ell+1}}{\Gamma(n-\ell-1)} \left(-\frac{d}{2tdt}\right)^n \int_t^x (s^2-t^2)^{n-\ell-2} sR(x,s)ds,$$

here n can be arbitrary integer satisfying $n > \ell+1$.

[9] V. V. Kravchenko, E. L. Shishkina and S. M. Torba, *On a series representation for integral kernels of transmutation operators for perturbed Bessel equations*, Math. Notes 104 (2018), 552–570.

Theorem

Let $q \in C^1[0,b]$. For $\ell = 0, 1, 2, \ldots$ the following formula for the integral kernel is valid

$$K(x,t) = \frac{\sqrt{\pi}\, t^{\ell+1}}{x^{2\ell+3}\Gamma\left(\ell+\frac{3}{2}\right)} \sum_{m=0}^{\infty} (-1)^{m+\ell+1} \times \frac{\Gamma\left(m+2\ell+\frac{5}{2}\right)}{\Gamma\left(m+\ell+\frac{3}{2}\right)} \beta_{m+\ell+1}(x) P_m^{(\ell+\frac{1}{2},\ell+1)}\left(1-2\frac{t^2}{x^2}\right),$$

where $P_m^{(\alpha,\beta)}$ stands for a Jacobi polynomial and the coefficients β_k are those from [Kravchenko, Torba, Castillo, 2018].
The series converges uniformly with respect to t on any segment $[\varepsilon, x-\varepsilon] \subset (0,x)$. Under the additional assumption that $q \in C^{2\ell+5}[0,b]$ the series converges absolutely and uniformly with respect to t on $[0,x]$.

Theorem

Let $q \in C^1[0,b]$. Then

$$K(x,t) = \frac{\sqrt{\pi}}{\Gamma(\ell+3/2)} \frac{t^{\ell+1}}{x(x^2-t^2)^{\ell+1}} \times \sum_{k=0}^{\infty} \frac{(-1)^k k!}{\Gamma(k-\ell)} \beta_k(x) P_k^{(\ell+1/2,-\ell-1)}\left(1-\frac{2t^2}{x^2}\right),$$

where $P_k^{(\ell+1/2,-\ell-1)}$ are polynomials given by the same formulas as the classical Jacobi polynomials.[a]
The series converges uniformly with respect to t on any segment $t \in [\varepsilon, x]$. Under the additional assumption that $q \in C^{2[\ell]+5}[0,b]$, the convergence is uniform with respect to t on $[0,x]$.

[a] Section 4.22, G. Szegö, "Orthogonal Polynomials, revised ed.", AMS, 1959

Form of the integral kernel R

Suppose $q \in C^{2p-1}[0,b]$. Than

$$R(x,t) = \sum_{k=1}^{p} A_k(x)\left(1-\frac{t^2}{x^2}\right)^{\ell+k} + \mathcal{R}_p(x,t), \qquad -x \le t \le x.$$

Moreover, if the potential q possesses holomorphic extension to a disk of radius $2xe\sqrt{1+|2\ell|}$, then[10]

$$\begin{aligned} R(x,t) &= \sum_{k=1}^{\infty} A_k(x)\left(1-\frac{t^2}{x^2}\right)^{\ell+k} \\ &= \left(1-\frac{t^2}{x^2}\right)^{\ell+1}\sum_{k=1}^{\infty} A_k(x)\left(1-\frac{t^2}{x^2}\right)^{k-1}. \end{aligned}$$

[10] H. Chébli, A. Fitouhi and M. M. Hamza, *Expansion in series of Bessel functions and transmutations for perturbed Bessel operators*, J. Math. Anal. Appl. 181 (1994), no. 3, 789–802.

Idea of improvement: to factor

$$\left(1-\frac{t^2}{x^2}\right)^{\ell+1}$$

from the integral kernel R and use it as a weight, i.e., to use the Fourier-Jacobi expansion.

Fourier-Jacobi expansion of the kernel $\tilde{R}$

Idea of improvement: to factor

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The integral kernel R can be represented as

$$R(x,t)=\left(1-\frac{t^2}{x^2}\right)^{\ell+1}\sum_{n=0}^{\infty}\frac{\tilde{\beta}_n(x)}{x}P_{2n}^{(\ell+1,\ell+1)}\left(\frac{t}{x}\right),$$

where $P_n^{(\alpha,\beta)}$ denotes the Jacobi polynomials.

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The main difficulty: we need smoothness properties of the function

$$\frac{R(x,t)}{\left(1-\frac{t^2}{x^2}\right)^{\ell+1}}$$

on the segment $t\in[-x,x]$.

Best Jacobi polynomial approximation

Let $\mathcal{P}_n$ be the set of algebraic polynomials of degree not greater than n.
Let $W_\alpha(x) = (1-x^2)^{\alpha/2}$, $x \in [-1,1]$, $\alpha > -1/2$.
We define as best weighted polynomial approximation of a function f such that $fW_\alpha \in L_2(-1,1)$ the quantity

$$E_n(W_\alpha; f) = \inf_{p_n \in \mathcal{P}_n} \|(f - p_n)W_\alpha\|_{L_2(-1,1)}, \qquad n = 0, 1, 2, \ldots$$

Let $\mathcal{P}_n$ be the set of algebraic polynomials of degree not greater than n.
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Let $S_k^{(\alpha)}$, $k = 0, 1, \ldots$ be the set of functions f satisfying

$$f^{(l)} W_{\alpha+l} \in L_2(-1,1), \qquad l = 0, 1, \ldots, k.$$

The following result holds [N. X. Ky, Acta Math. Hung. 27 (1976)]

Theorem

Let $f \in S_k^{(\alpha)}$ for some $k \in \mathbb{N}_0$. Then

$$E_{n+k}(W_\alpha; f) \le \frac{c(\alpha,k)}{n^k}\omega\left(W_{\alpha+k}; f^{(k)}; \frac{1}{n}\right) \le \frac{c_1(\alpha,k)}{n^k}\|W_{\alpha+k}f^{(k)}\|_{L_2(-1,1)},$$

Smoothness of R

Recall that the integral kernel R is the Fourier transform of

$$y_\ell(\omega, x) - d_\ell(\omega) b_\ell(\omega x)$$

which is an entire function of exponential type x and satisfies

$$|y_\ell(\omega, x) - d_\ell(\omega) b_\ell(\omega x)| \le \frac{C}{|\omega|^{\ell+2}}, \qquad \omega \in \mathbb{R}.$$

[11] Theorems 1 and 3 from F. Móricz, *Absolutely convergent Fourier integrals and classical function spaces*, Arch. Math. 91 (2008), 49–62.

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Hence $\operatorname{supp} R(x, \cdot) \subset [-x, x]$ and[11] (let $\ell \notin \mathbb{Z}$)

$$R(x, \cdot) \in \operatorname{Lip}(\ell + 1; \mathbb{R})$$

Moreover,

$$\partial_t^k R(x, \cdot) \in \operatorname{Lip}(\ell + 1 - k; \mathbb{R}), \quad k = 1, 2, \dots, [\ell + 1].$$

[11] Theorems 1 and 3 from F. Móricz, *Absolutely convergent Fourier integrals and classical function spaces*, Arch. Math. 91 (2008), 49–62.

By $[[x]]$ we denote the largest integer smaller than x, and let $\{\{x\}\} := x - [[x]]$. Then $\{\{x\}\} \in (0,1]$.

Proposition

Let $x > 0$ be fixed and $q \in W_1^{2p-1}[0,b]$. Then

1. *there exist functions $\{r_m\}_{m=0}^{[[\ell+1+p]]}$, bounded on $[-x,x]$ and continuous on $(-x,x)$, such that*

$$\partial_t^m R(x,t) = (x^2 - t^2)^{\ell+1+p-m} r_m(t), \qquad t \in [-x,x].$$

2. *The last derivative satisfies*

$$\left|\partial_t^{[[\ell+1+p]]} R(x,t)\right| \le c(x^2 - t^2)^{\{\{\ell+1+p\}\}} \qquad t \in [-x,x].$$

Let $x > 0$ be fixed. Consider the functions

$$g(z) := R(x, zx) \qquad \text{and} \qquad h(z) := \frac{g(z)}{(1-z^2)^{\ell+1}}, \qquad z \in (-1, 1).$$

Note that

$$h^{(m)}(z) = \sum_{k=0}^{m} \frac{g^{(m-k)}(z) \cdot p_k(z)}{(1-z^2)^{\ell+1+k}},$$

where p_k are some polynomials in z.

Proposition

Suppose that $q \in W_1^{2p-1}[0, b]$ *for some* $p \in \mathbb{N}$. *Then*

$$h \in S^{(\ell+1)}_{[[\ell+p+3/2]]}.$$

Theorem

Let $q \in W_1^{2p-1}[0,b]$ and $x > 0$ be fixed. There exists a constant $C = C(R,x,\ell)$ such that

$$\left\| \frac{R(x,t)}{\left(1-\frac{t^2}{x^2}\right)^{\frac{\ell+1}{2}}} - \left(1-\frac{t^2}{x^2}\right)^{\frac{\ell+1}{2}} \sum_{k=0}^{N} c_{2k} P_{2k}^{(\ell+1,\ell+1)}\left(\frac{t}{x}\right) \right\|_{L_2(-x,x)} \le \frac{C}{N^{\ell+p+1}},$$

for all $2N > \ell + p + 3/2$.

Corollary

Let $q \in W_1^{2p-1}[0,b]$. Then there exists a constant $C = C(R,x,\ell)$ such that

$$\sum_{n=N+1}^{\infty} \frac{|\tilde{\beta}_n(x)|^2}{n} \le \frac{C}{N^{2\ell+2p+2}}, \qquad 2N > \ell + p + 3/2.$$

Let $m := [\ell]$, $\lambda := \{\ell\}$, so $\ell = m + \lambda$. We have

$$K(x,t) = \frac{4\sqrt{\pi}t^{\ell+1}}{\Gamma(\ell+3/2)}\left(-\frac{d}{2tdt}\right)^{m+3}\sum_{n=0}^{\infty}\frac{\tilde{\beta}_n(x)}{\Gamma(2-\lambda)x^{2\ell+3}}$$
$$\times\int_t^x (s^2-t^2)^{1-\lambda}s(x^2-s^2)^{\ell+1}P_{2n}^{(\ell+1,\ell+1)}\left(\frac{s}{x}\right)\,ds.$$

The integral can be calculated [12]

$$\int_t^x (s^2-t^2)^{1-\lambda}s(x^2-s^2)^{\ell+1}C_{2n}^{\ell+3/2}\left(\frac{s}{x}\right)\,ds = \frac{x^{2m+6}(2\ell+3)_{2n}\Gamma(\ell+2)}{2(2n)!}$$
$$\times(-1)^n\sum_{k=0}^{m+n+3}\frac{(\lambda-1)_k}{k!}\frac{\Gamma(2-\lambda-k)}{\Gamma(m+n+4-k)}\frac{\Gamma(k+n+\lambda-3/2)}{\Gamma(k+\lambda-3/2)}\left(\frac{t}{x}\right)^{2k}.$$

[12] (2.21.1.4) from A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, “Integrals and series. Vol. 2. Special functions”, 1986, 750 pp.

Continued

For the derivative of order $m+3$ we obtain

$$\left(\frac{d}{dt^2}\right)^{m+3} \sum_{k=0}^{m+n+3} \frac{(-1)^k}{k!(m+n+3-k)!} \frac{\Gamma(k+n+\lambda-3/2)}{\Gamma(k+\lambda-3/2)} \left(\frac{t}{x}\right)^{2k}$$
$$= \frac{(-1)^{m+n+3}}{x^{2m+6}} P_n^{(0,\ell+1/2)}\left(2\frac{t^2}{x^2}-1\right)$$
$$= \frac{(-1)^{m+1}}{x^{2m+6}} P_n^{(\ell+1/2,0)}\left(1-2\frac{t^2}{x^2}\right).$$

Finally we obtain

$$K(x,t) = \frac{2\sqrt{\pi}t^{\ell+1}}{x^{2\ell+3}\Gamma(\ell+3/2)}\sum_{n=0}^{\infty}\frac{(-1)^n\tilde{\beta}_n(x)\Gamma(\ell+2n+2)}{(2n)!}P_n^{(\ell+1/2,0)}\left(1-\frac{2t^2}{x^2}\right)$$

$$= \frac{t^{\ell+1}}{x^{\ell+2}}\sum_{n=0}^{\infty}\beta_n(x)P_n^{(\ell+1/2,0)}\left(1-\frac{2t^2}{x^2}\right),$$

where we denoted

$$\beta_n(x) = (-1)^n\frac{2\sqrt{\pi}\Gamma(\ell+2n+2)}{x^{\ell+1}\Gamma(\ell+3/2)(2n)!}\tilde{\beta}_n(x).$$

Problem with convergency here:

the coefficients

$$\frac{\Gamma(\ell+2n+2)}{(2n)!} \sim n^{\ell+1}.$$

First we need the following lemma.

Lemma

$$\int_0^x t^{\ell+3/2} P_n^{(\ell+1/2,0)}\left(1-2\frac{t^2}{x^2}\right) J_{\ell+1/2}(\omega t)\,dt = \frac{x^{\ell+3/2}}{\omega} J_{\ell+2n+3/2}(\omega x).$$

Theorem

Let $q \in W_1^{2p-1}[0,b]$. Then the regular solution $u(\omega,x)$ has the following representation

$$u(\omega,x) = \omega x j_\ell(\omega x) + \sum_{n=0}^{\infty} \beta_n(x) j_{\ell+2n+1}(\omega x).$$

The series converges absolutely and uniformly with respect to ω on any compact subset of the complex plane.

Theorem

Denote by

$$u_N(\omega, x) = \omega x j_\ell(\omega x) + \sum_{n=0}^{N} \beta_n(x) j_{\ell+2n+1}(\omega x)$$

an approximate solution. Then the following estimate holds uniformly for $\omega \in \mathbb{R}$

$$|u(\omega, x) - u_N(\omega, x)| \leq \frac{c(x)}{N^p}, \qquad 2N > \ell + p + 2,$$

here p is the parameter of smoothness of the potential q.

Substituting the improved NSBF representation into equation we get

$$\frac{1}{(2\ell+3)x^{\ell+1}} L\left[x^{\ell+1}\beta_0(x)\right] = -q(x),$$

$$\frac{1}{(4n+2\ell+3)x^{2n+\ell+1}} L\left[x^{2n+\ell+1}\beta_n(x)\right] = -\frac{x^{2n+\ell}}{4n+2\ell-1} L\left[\frac{\beta_{n-1}(x)}{x^{2n+\ell}}\right], \quad n \geq 1$$

Or

$$\beta_0(x) = (2\ell+3)\left(\frac{u_0(x)}{x^{\ell+1}} - 1\right),$$

$$\eta_n(x) = \int_0^x \left(t u_0'(t) + (2n+\ell)u_0(t)\right) t^{2n+\ell-1}\beta_{n-1}(t)\,dt,$$

$$\theta_n(x) = \int_0^x \frac{1}{u_0^2(t)}\left(\eta_n(t) - t^{2n+\ell}\beta_{n-1}(t)u_0(t)\right)dt, \qquad n \geq 1,$$

and finally

$$\beta_n(x) = -\frac{4n+2\ell+3}{4n+2\ell-1}\left[\beta_{n-1}(x) + \frac{2(4n+2\ell+1)u_0(x)\theta_n(x)}{x^{2n+\ell+1}}\right].$$

THANKS!